Tohoku Math. J.
55 (2003), 361–374

ON THE REAL SECONDARY CLASSES OF TRANSVERSELY HOLOMORPHIC FOLIATIONS II

TARO ASUKE

(Received July 16, 2001, revised May 30, 2002)

Abstract. An algorithm to find a basis of the space of complex secondary classes of transversely holomorphic foliations is given. The mapping which relates the real secondary classes to the complex secondary classes is completely described when the complex codimension of the foliation is either two or three. Finally, it is shown that the image of real secondary classes under this mapping is naturally reduced under a certain condition.

Introduction. Associated with transversely holomorphic foliations, there exist complex secondary classes and real secondary classes. The spaces of these secondary classes are denoted by $H^*(\mathrm{WU}_q)$ and $H^*(\mathrm{WO}_{2q})$, respectively, where q is the complex codimension of foliations. Forgetting transverse holomorphic structures determines a natural mapping $[\lambda]$ from $H^*(\mathrm{WO}_{2q})$ to $H^*(\mathrm{WU}_q)$ [2]. Roughly speaking, the mapping $[\lambda]$ divides the elements of $H^*(\mathrm{WO}_{2q})$ and $H^*(\mathrm{WU}_q)$, respectively, into two parts. Namely, the kernel of $[\lambda]$ can be considered as obstructions to foliations being transversely holomorphic, and the coimage of $[\lambda]$ consists of real secondary classes which still make sense as characteristic classes of transversely holomorphic foliations. On the other hand, the image of $[\lambda]$ consists of complex secondary classes which are in fact real secondary classes, and the cokernel of $[\lambda]$ consists of purely complex secondary characteristic classes.

The mapping $[\lambda]$ is not yet well-understood except for the case where $q = 1$ [2]. One way to study the mapping $[\lambda]$ is to know the space $H^*(\mathrm{WU}_q)$. We first give an algorithm to find a basis of $H^*(\mathrm{WU}_q)$ (Theorem 1.7) and compute it in the cases where $q = 2$ or $q = 3$. Using these bases, the mapping $[\lambda]$ is completely determined in the corresponding cases (Theorems 1.8 and 1.9).

Since the foliations under consideration are of complex codimension q, it is natural to expect that the image of $H^*(\mathrm{WO}_{2q})$ in $H^*(\mathrm{WU}_q)$ can be described only in terms of h_i and c_j with $i \le q$ and $j \le q$. Regarding this, we introduce certain classes $[\tilde{h}_i]$ in $H^*(\mathrm{WU}_q)$ and state a sufficient condition in terms of these classes (Definition 3.4 and Proposition 3.8). Examples show that these classes are non-trivial in general.

This paper is based on a part of the author's thesis [1]. He would like to express his gratitude to Professors T. Tsuboi, S. Morita and D. Lehmann for their helpful comments and suggestions. He is also grateful to the referee for his comments on the exposition.

2000 *Mathematics Subject Classification.* Primary 58H10; Secondary 32S65, 53C12.
Key words and phrases. Foliations, Transverse holomorphic structure, Characteristic classes.
Supported by the JSPS Postdoctoral Fellowships for Research Abroad (1999–2000).
Supported by the Ministry of Education, Culture, Sports, Science and Technology, Grant No. 13740042.

1. Definitions and statements of the main results. First of all, we briefly recall relevant definitions. We refer the readers to [5, 11, 2] for more details. Let $\boldsymbol{C}[v_1, \dots, v_q]$ be the polynomial ring generated by $v_1, \dots, v_q$, where we consider the degree of v_i as $2i$. Let I_q be the ideal generated by monomials of degree greater than $2q$, and set $\boldsymbol{C}_q[v_1, \dots, v_q] = \boldsymbol{C}[v_1, \dots, v_q]/I_q$. $\boldsymbol{C}[\bar{v}_1, \dots, \bar{v}_q]$ is defined simply by replacing v_i with $\bar{v}_i$. Similarly, $\boldsymbol{R}_{2q}[c_1, \dots, c_{2q}]$ is defined, namely, $\boldsymbol{R}_{2q}[c_1, \dots, c_{2q}] = \boldsymbol{R}[c_1, \dots, c_{2q}]/I'_{2q}$, where we consider the degree of c_i as $2i$ and I'_{2q} is the ideal generated by monomials of degree greater than $4q$.

DEFINITION 1.1. We define graded differential algebras $\mathrm{WU}_q^{(l)}$, $l \le q$, and $\mathrm{WO}_{2q}^{(m)}$, $m \le 2q$, as follows:

$$\mathrm{WU}_q^{(l)} = \boldsymbol{C}_q[v_1, \dots, v_q] \otimes \boldsymbol{C}_q[\bar{v}_1, \dots, \bar{v}_q] \otimes \bigwedge[\tilde{u}_1, \dots, \tilde{u}_l]\,,$$
$$\mathrm{WO}_{2q}^{(m)} = \boldsymbol{R}_{2q}[c_1, \dots, c_{2q}] \otimes \bigwedge[h_1, h_3, \dots, h_{m'}]\,,$$

where m' denotes the greatest odd integer not greater than m. We denote $\mathrm{WU}_q^{(q)}$ by WU_q and $\mathrm{WO}_{2q}^{(2q)}$ by WO_{2q}, respectively. These algebras are equipped with differentials determined by requiring $d\tilde{u}_i = v_i - \bar{v}_i$, $dv_i = d\bar{v}_i = 0$, $dh_i = c_i$ and $dc_i = 0$. The elements $\tilde{u}_i$ and h_i are considered to be of degree $2i - 1$. Finally, we set $\mathrm{WU}_q^{(l)} = \mathrm{WU}_q$ if $l > q$.

The natural mapping $[\lambda]$ in the introduction is induced by the following mapping λ.

DEFINITION 1.2 [2]. Let λ be the mapping from WO_{2q} to WU_q defined by

$$\lambda(c_k) = (\sqrt{-1})^k \sum_{j=0}^{k} (-1)^j v_{k-j}\bar{v}_j\,,$$
$$\lambda(h_{2k+1}) = \frac{(-1)^k}{2}\sqrt{-1} \sum_{j=0}^{2k+1} (-1)^j \tilde{u}_{2k-j+1}(v_j + \bar{v}_j)\,.$$

We denote by $[\lambda]$ the mapping induced on the cohomology.

If in addition the elements h_{2k}, u_i and $\bar{u}_i$ (such that $dh_{2k} = c_{2k}$, $du_i = v_i$, $d\bar{u}_i = \bar{v}_i$, $u_i - \bar{u}_i = \tilde{u}_i$) are well-defined, we set

$$\lambda(h_{2k}) = (-1)^k \frac{1}{2} \sum_{j=0}^{2k} (-1)^j (u_{2k-j}\bar{v}_j + \bar{u}_j v_{2k-j})\,,$$

where v_0, $\bar{v}_0$ are considered as 2.

The mapping λ maps $\mathrm{WO}_{2q}^{(l)}$ to $\mathrm{WU}_q^{(l)}$ and commutes with the differentials and natural inclusions.

DEFINITION 1.3. Let α be a cocycle in WO_{2q}. We denote by $[\alpha]$ the cohomology class represented by α. Its image by $[\lambda]$ is denoted by $[\alpha]_\lambda$. For a cocycle β in WU_q, we also denote by $[\beta]$ the cohomology class defined by β. Thus $[\alpha]_\lambda = [\lambda(\alpha)]$.

We now recall the Vey basis [9]. First, let $I = \{i_1, \ldots, i_m\}$ be an index set which consists of odd integers such that $1 \leq i_1 < \cdots < i_m < 2q$, and let $J = (j_1, \ldots, j_{2q})$ be an index set which consists of nonnegative integers. We set $|J| = j_1 + 2j_2 + \cdots + (2q)j_{2q}$. Then the Vey basis of $H^*(\mathrm{WO}_{2q})$ is given by

$$\begin{aligned}&\{[c_J]\,;\ 1 \leq |J| \leq 2q,\ j_k = 0 \text{ for all odd integers } k\}\\ &\quad \cup \{[h_I c_J]\,;\ |J| \leq 2q,\ i_1 + |J| > 2q,\ i_1 \leq k \text{ if } k \text{ is odd and } j_k > 0\}\ ,\end{aligned}$$

where $h_I c_J = c_1^{j_1} \cdots c_{2q}^{j_{2q}} \otimes h_{i_1} \cdots h_{i_l}$. The cocycles of the form $h_I c_J$ fulfilling the above conditions are called Vey cocycles.

DEFINITION 1.4. Suppose that $I = \{i_1, \ldots, i_m\}$ and $I' = \{i'_1, \ldots, i'_m\}$. We set $i_\infty = i_m$, and say $I < I'$ if $i_\infty < i'_\infty$.

DEFINITION 1.5. Let $J = (j_1, \ldots, j_{2q})$ and $K = (k_1, \ldots, k_{2q})$ be as above.

1) We denote respectively by j_0 and j_∞ the smallest and largest number l such that $j_l \neq 0$. We say $J \leq p$ (resp. $J \geq p$) if $j_\infty \leq p$ (resp. $j_0 \geq p$).

2) We say $J < K$ if there is a positive integer l such that $j_m = k_m$ if $m > l$ and $j_l < k_l$.

3) We set $J+K = (j_1+k_1, \ldots, j_{2q}+k_{2q})$ and call it the sum of J and K. Conversely, if $L = J + K$, then the pair J and K is called as a decomposition of L.

DEFINITION 1.6. We define differential operators ∂_l, $l = 1, \ldots, q$, by requiring

$$\partial_l(\tilde{u}_i) = \begin{cases} v_l - \bar{v}_l & \text{if } l = i\,,\\ 0 & \text{otherwise}\,.\end{cases}$$

∂_l induces a differential operator $[\partial_l]$ on $H^*(\mathrm{WU}_q^{(l-1)}) \oplus \tilde{u}_l \wedge H^*(\mathrm{WU}_q^{(l-1)})$ in an obvious way. Clearly, $[\partial_l] \circ [\partial_l] = 0$ and we can consider the $[\partial_i]$-cohomology.

With these preparations, we show the following.

THEOREM 1.7. *The space* $H^*(\mathrm{WU}_q)$ *is calculated by taking consecutively the* $[\partial_i]$-*cohomology from* $[\partial_1]$ *to* $[\partial_q]$.

PROOF. We first set $A^{(l)} = \tilde{u}_l \wedge \mathrm{WU}_q^{(l-1)}$. Then we have an exact sequence $0 \to \mathrm{WU}_q^{(l-1)} \to \mathrm{WU}_q^{(l)} \to A^{(l)} \to 0$. Consider the associated long exact sequence

$$\cdots \to H^*(\mathrm{WU}_q^{(l-1)}) \to H^*(\mathrm{WU}_q^{(l)}) \to H^*(A^{(l)}) \xrightarrow{\tau} H^{*+1}(\mathrm{WU}_q^{(l-1)}) \to \cdots\,,$$

where τ is in fact equal to $[\partial_l]$ defined above. To see this, let $\tilde{u}_l\omega \in A^{(l)}$. The equation $d(\tilde{u}_l\omega) = (v_l - \bar{v}_l)\omega - \tilde{u}_l(\partial_1 + \cdots + \partial_{l-1})\omega$ implies that $d_{A^{(l)}}(\tilde{u}_l\omega) = (\partial_1 + \cdots + \partial_{l-1})(\tilde{u}_l\omega)$. Thus $H^*(A^{(l)}) = \tilde{u}_l \wedge H^*(\mathrm{WU}_q^{(l-1)})$. On the other hand, if $d_{A^{(l)}}(\tilde{u}_l\omega) = 0$, then $\tau(\tilde{u}_l\omega)$ is given by the class represented by $\partial_l(\tilde{u}_l\omega)$ because $(\partial_1 + \cdots + \partial_{l-1})(\tilde{u}_l\omega) = 0$. Consequently, $H^*(\mathrm{WU}_q)$ is calculated by computing consecutively the above exact sequences with $H^*(A^{(l)})$ replaced by $\tilde{u}_l \wedge H^*(\mathrm{WU}_q^{(l-1)})$. This completes the proof. □

We now give a basis of $H^*(\mathrm{WU}_q)$ and a description of $[\lambda]$ respectively in the case where $q = 2$ or $q = 3$. In what follows, there appear several tables which contain numbers and cocycles. Such numbers stand for the degree of the cocycles in the same rows.

THEOREM 1.8. 1) *The classes represented by the following cocycles form a basis of* $H^*(\mathrm{WU}_2)$.

2	$(v_1 + \bar{v}_1)$
4	$(v_1^2 + v_1\bar{v}_1 + \bar{v}_1^2), (v_2 + \bar{v}_2)$
5	$\tilde{u}_1(v_1^2 + v_1\bar{v}_1 + \bar{v}_1^2), \tilde{u}_2(v_1 + \bar{v}_1) + \tilde{u}_1(v_2 + \bar{v}_2)$
7	$\tilde{u}_1 v_1\bar{v}_1(v_1 + \bar{v}_1), \tilde{u}_2(v_1^2 + v_1\bar{v}_1 + \bar{v}_1^2) + \tilde{u}_1(v_1\bar{v}_2 + v_2\bar{v}_1), \tilde{u}_2(v_2 + \bar{v}_2)$
9	$\tilde{u}_1 v_1^2\bar{v}_1^2, \tilde{u}_1(v_1^2\bar{v}_2 + v_2\bar{v}_1^2), \tilde{u}_1 v_2\bar{v}_2$
10	$\tilde{u}_1\tilde{u}_2 v_1\bar{v}_1(v_1 + \bar{v}_1)$
11	$\tilde{u}_2 v_2\bar{v}_2$
12	$\tilde{u}_1\tilde{u}_2 v_1^2\bar{v}_1^2, \tilde{u}_1\tilde{u}_2 v_1^2\bar{v}_2, \tilde{u}_1\tilde{u}_2 v_2\bar{v}_1^2, \tilde{u}_1\tilde{u}_2 v_2\bar{v}_2$

2) *As a basis of the image of* $H^*(\mathrm{WO}_4)$ *in* $H^*(\mathrm{WU}_2)$, *we can take the following classes*:

$$[c_2]_\lambda, [h_1c_2^2]_\lambda, [h_1c_1^2c_2]_\lambda, [h_1c_1^4]_\lambda .$$

3) *The kernel of the mapping* $[\lambda]$ *is spanned by the following classes*:

$$[c_2]^2, \quad [c_4],$$
$$[h_3c_2] - \frac{1}{2}[h_1c_1c_3],$$
$$[h_1c_4] - \frac{1}{2}[h_1c_2^2] + \frac{1}{12}[h_1c_1^4], \quad [h_1c_1c_3] - [h_1c_1^2c_2] + \frac{1}{3}[h_1c_1^4],$$
$$[h_3c_J], \quad \textit{where} \quad |J| \geq 3,$$
$$[h_1h_3c_J], \quad \textit{where} \quad |J| \geq 4.$$

4) *The image is equal to* $\langle[\bar{v}_1]^2 - 2[\bar{v}_2]\rangle \oplus H^9(\mathrm{WU}_2)$, *where* $\langle[\bar{v}_1]^2 - 2[\bar{v}_2]\rangle$ *denotes the linear subspace spanned by* $[\bar{v}_1]^2 - 2[\bar{v}_2]$. *In particular*, *the subspace spanned by the classes* $[h_1c_1^4]$, $[h_1c_1^2c_2]$ *and* $[h_1c_2^2]$ *in* $H^9(\mathrm{WO}_4)$ *is mapped isomorphically to* $H^9(\mathrm{WU}_2)$.

5) *The cokernel consists of the secondary classes of* $H^*(\mathrm{WU}_2)$ *which are not of degree* 9 *and the subspace spanned by the classes* $[\bar{v}_1]$, $[\bar{v}_1]^2 + 2[\bar{v}_2]$.

THEOREM 1.9. 1) *The classes represented by the following cocycles form a basis of* $H^*(\mathrm{WU}_3)$.

	$\bar v_1, \bar v_1^2, \bar v_2, \bar v_1^3, \bar v_1\bar v_2, \bar v_3$
7	$\tilde u_1(v_1^3+v_1^2\bar v_1+v_1\bar v_1^2+\bar v_1^3), \tilde u_2 v_1^2+\tilde u_1 v_1\bar v_2+\tilde u_1\bar v_1\bar v_2, \tilde u_2(v_2+\bar v_2),$ $\tilde u_3 v_1+\tilde u_1\bar v_3$
9	$\tilde u_1(v_1^3\bar v_1+v_1^2\bar v_1^2+v_1\bar v_1^3), \tilde u_2 v_1^3+\tilde u_1 v_1^2\bar v_2+\tilde u_1 v_1\bar v_1\bar v_2,$ $\tilde u_2(v_1v_2+v_1\bar v_2), \tilde u_3 v_1^2+\tilde u_1 v_1\bar v_3, \tilde u_3 v_2+\tilde u_2\bar v_3$
11	$\tilde u_1(v_1^3\bar v_1^2+v_1^2\bar v_1^3), \tilde u_1(v_1v_2\bar v_1^2+v_2\bar v_1^3), \tilde u_1(v_1+\bar v_1)v_2\bar v_2,$ $\tilde u_2 v_2\bar v_2, \tilde u_3 v_1^3+\tilde u_1 v_1^2\bar v_3, \tilde u_3 v_1v_2+\tilde u_1 v_2\bar v_3, \tilde u_3(v_3+\bar v_3)$
13	$\tilde u_1 v_1^3\bar v_1^3, \tilde u_1 v_1v_2\bar v_1^3, \tilde u_1 v_1v_2\bar v_1\bar v_2, \tilde u_1 v_1v_2\bar v_3, \tilde u_1 v_3\bar v_1^3, \tilde u_1 v_3\bar v_3,$ $\tilde u_2 v_1v_2\bar v_2, \tilde u_2 v_2\bar v_3$
14	$\tilde u_1\tilde u_2(v_1^3\bar v_1^2+v_1^2\bar v_1^3), \tilde u_1\tilde u_2(v_1v_2\bar v_1^2+v_2\bar v_1^3), \tilde u_1\tilde u_2(v_1^3\bar v_2+v_1^2\bar v_1\bar v_2),$ $\tilde u_1\tilde u_2(v_1+\bar v_1)v_2\bar v_2, \tilde u_1\tilde u_3(v_1^3\bar v_1+v_1^2\bar v_1^2+v_1\bar v_1^3),$ $\tilde u_2\tilde u_3 v_1^3+\tilde u_1\tilde u_3 v_1^2\bar v_2+\tilde u_1\tilde u_3 v_1\bar v_1\bar v_2-\tilde u_1\tilde u_2 v_1^2\bar v_3, \tilde u_2\tilde u_3(v_1v_2+v_1\bar v_2)-\tilde u_1\tilde u_2 v_2\bar v_3$
15	$\tilde u_2 v_3\bar v_3$
16	$\tilde u_1\tilde u_2 v_1^3\bar v_1^3, \tilde u_1\tilde u_2 v_1^3\bar v_1\bar v_2, \tilde u_1\tilde u_2 v_1^3\bar v_3, \tilde u_1\tilde u_2 v_1v_2\bar v_1^3, \tilde u_1\tilde u_2 v_1v_2\bar v_1\bar v_2, \tilde u_1\tilde u_2 v_1v_2\bar v_3,$ $\tilde u_1\tilde u_2 v_3\bar v_1^3, \tilde u_1\tilde u_2 v_3\bar v_1\bar v_2, \tilde u_1\tilde u_2 v_3\bar v_3, \tilde u_1\tilde u_3(v_1^3\bar v_1^2+v_1^2\bar v_1^3),$ $\tilde u_1\tilde u_3(v_1v_2\bar v_1^2+v_2\bar v_1^3), \tilde u_1\tilde u_3(v_1+\bar v_1)v_2\bar v_2, \tilde u_2\tilde u_3 v_2\bar v_2$
17	$\tilde u_3 v_3\bar v_3$
18	$\tilde u_1\tilde u_3 v_1^3\bar v_1^3, \tilde u_1\tilde u_3 v_1^3\bar v_3, \tilde u_1\tilde u_3 v_1v_2\bar v_1^3, \tilde u_1\tilde u_3 v_1v_2\bar v_1\bar v_2, \tilde u_1\tilde u_3 v_1v_2\bar v_3,$ $\tilde u_1\tilde u_3 v_3\bar v_1^3, \tilde u_1\tilde u_3 v_3\bar v_1\bar v_2, \tilde u_1\tilde u_3 v_3\bar v_3, \tilde u_2\tilde u_3 v_1v_2\bar v_2, \tilde u_2\tilde u_3 v_3\bar v_2, \tilde u_2\tilde u_3 v_2\bar v_3$
19	$\tilde u_1\tilde u_2\tilde u_3(v_1^3\bar v_1^2+v_1^2\bar v_1^3), \tilde u_1\tilde u_2\tilde u_3(v_1v_2\bar v_1^2+v_2\bar v_1^3), \tilde u_1\tilde u_2\tilde u_3(v_1^3\bar v_2+v_1^2\bar v_1\bar v_2),$ $\tilde u_1\tilde u_2\tilde u_3(v_1+\bar v_1)v_2\bar v_2$
20	$\tilde u_2\tilde u_3 v_3\bar v_3$
21	$\tilde u_1\tilde u_2\tilde u_3 v_1^3\bar v_1^3, \tilde u_1\tilde u_2\tilde u_3 v_1^3\bar v_1\bar v_2, \tilde u_1\tilde u_2\tilde u_3 v_1^3\bar v_3, \tilde u_1\tilde u_2\tilde u_3 v_1v_2\bar v_1^3, \tilde u_1\tilde u_2\tilde u_3 v_1v_2\bar v_1\bar v_2,$ $\tilde u_1\tilde u_2\tilde u_3 v_1v_2\bar v_3, \tilde u_1\tilde u_2\tilde u_3 v_3\bar v_1^3, \tilde u_1\tilde u_2\tilde u_3 v_3\bar v_1\bar v_2, \tilde u_1\tilde u_2\tilde u_3 v_3\bar v_3$

2) *As a basis of the image of* $H^*(\mathrm{WO}_6)$ *in* $H^*(\mathrm{WU}_3)$, *we can take the following classes*:

4	$[c_2]_\lambda$
13	$[h_1c_3^2]_\lambda, [h_1c_1c_2c_3]_\lambda, [h_1c_1^3c_3]_\lambda, [h_1c_1^4c_2]_\lambda, [h_1c_1^2c_2^2]_\lambda, [h_1c_1^6]_\lambda, [h_3c_2^2]_\lambda$
17	$[h_3c_3^2]_\lambda$
18	$[h_1h_3c_3^2]_\lambda, [h_1h_3c_1c_2c_3]_\lambda, [h_1h_3c_1^3c_3]_\lambda, [h_1h_3c_1^4c_2]_\lambda, [h_1h_3c_1^2c_2^2]_\lambda, [h_1h_3c_1^6]_\lambda$

3) *The kernel is spanned by the following classes*, *namely*,

$$[h_1c_2^3]-\frac{1}{8}[h_1c_1^6]+\frac{3}{4}[h_1c_1^4c_2]-\frac{3}{2}[h_1c_1^2c_2^2]\,,$$

$$[h_1c_2c_4]-\frac{1}{16}[h_1c_1^6]-[h_1c_1c_2c_3]+\frac{1}{4}[h_1c_1^2c_2^2]+\frac{1}{8}[h_1c_1^4c_2]\,,$$

$$[h_1c_1^2c_4] - \frac{1}{4}[h_1c_1^6] + [h_1c_1^4c_2] - [h_1c_1^3c_3] - \frac{1}{2}[h_1c_1^2c_2^2]\,,$$
$$[h_1c_1c_5] - [h_1c_1c_2c_3] - \frac{1}{20}[h_1c_1^6] + \frac{1}{2}[h_1c_1^2c_2^2]\,,$$
$$[h_1c_6] + \frac{1}{80}[h_1c_1^6] - \frac{1}{8}[h_1c_1^4c_2] + \frac{1}{4}[h_1c_1^2c_2^2] - \frac{1}{2}[h_1c_3^2]\,,$$
$$[h_3c_4] - \frac{1}{4}[h_1c_1^3c_3] + [h_1c_1c_2c_3] - [h_1c_3^2] - \frac{1}{2}[h_3c_2^2]\,,$$
$$[h_5c_2] - \frac{1}{2}[h_1c_1c_5]\,,$$
$$[h_3c_5],\ [h_3c_2c_3]\,,$$
$$[h_3c_6] - \frac{1}{2}[h_3c_3^2]\,,$$
$$[h_5c_4],\ [h_5c_2^2],\ [h_3c_2^3],\ [h_3c_2c_4]\,,$$
$$[h_1h_3c_2^3] - \frac{1}{8}[h_1h_3c_1^6] + \frac{3}{4}[h_1h_3c_1^4c_2] - \frac{3}{2}[h_1h_3c_1^2c_2^2]\,,$$
$$[h_1h_3c_2c_4] - \frac{1}{16}[h_1h_3c_1^6] - [h_1h_3c_1c_2c_3] + \frac{1}{4}[h_1h_3c_1^2c_2^2] + \frac{1}{8}[h_1h_3c_1^4c_2]\,,$$
$$[h_1h_3c_1^2c_4] - \frac{1}{4}[h_1h_3c_1^6] + [h_1h_3c_1^4c_2] - [h_1h_3c_1^3c_3] - \frac{1}{2}[h_1h_3c_1^2c_2^2]\,,$$
$$[h_1h_3c_1c_5] - [h_1h_3c_1c_2c_3] - \frac{1}{20}[h_1h_3c_1^6] + \frac{1}{2}[h_1h_3c_1^2c_2^2]\,,$$
$$[h_1h_3c_6] + \frac{1}{80}[h_1h_3c_1^6] - \frac{1}{8}[h_1h_3c_1^4c_2] + \frac{1}{4}[h_1h_3c_1^2c_2^2] - \frac{1}{2}[h_1h_3c_3^2]\,,$$
$$[h_5c_5]\,,$$

the secondary classes of degree greater than 20, *and the Pontrjagin classes other than* $[c_2]$.

4) *The image is described as follows*:

i) *The only Chern class in the image is* $[\bar{v}_1]^2 - 2[\bar{v}_2]$.

ii) *The image of the secondary classes is contained in the subspace* $H^{13}(\mathrm{WU}_3) \oplus H^{17}(\mathrm{WU}_3) \oplus H^{18}(\mathrm{WU}_3)$, *more precisely*,

ii-a) *the subspace of* $H^{13}(\mathrm{WO}_6)$ *spanned by the classes*

$$[h_1c_1^6]\,,\quad [h_1c_1^4c_2]\,,\quad [h_1c_1^3c_3]\,,\quad [h_1c_1c_2c_3]\,,\quad [h_1c_3^2]\,,\quad [h_1c_1^2c_2^2]$$

is mapped to the subspace of $H^{13}(\mathrm{WU}_3)$ *spanned by the classes*

$$[\tilde{u}_1v_1^3\bar{v}_1^3]\,,\quad [\tilde{u}_1v_1v_2\bar{v}_1^3]\,,\quad [\tilde{u}_1v_1v_2\bar{v}_1\bar{v}_2]\,,\quad [\tilde{u}_1v_1v_2\bar{v}_3]\,,\quad [\tilde{u}_1v_3\bar{v}_1^3]\,,\quad [\tilde{u}_1v_3\bar{v}_3]\,.$$

The class $[h_3c_2^2]$ *is mapped to the class* $[\tilde{u}_2v_1v_2\bar{v}_2] - [\tilde{u}_2v_2\bar{v}_3]$ *modulo the above subspace.*

ii-b) *The class* $[h_3c_3^2]$, *of degree* 17, *is mapped to the class* $[\tilde{u}_3v_3\bar{v}_3]$.

ii-c) *The subspace spanned by the classes*

$$[h_1h_3c_1^6]\,,\quad [h_1h_3c_1c_2c_3]\,,\quad [h_1h_3c_1^3c_3]\,,\quad [h_1h_3c_3^2]\,,\quad [h_1h_3c_1^4c_2]\,,\quad [h_1h_3c_1^2c_2^2]\,,$$

which are of degree 18, *is mapped to the subspace spanned by the classes*

$$[\tilde{u}_1\tilde{u}_3 v_1^3\bar{v}_1^3]\,, \quad [\tilde{u}_1\tilde{u}_3(v_1v_2\bar{v}_3 + v_3\bar{v}_1\bar{v}_2)]\,, \quad [\tilde{u}_1\tilde{u}_3(v_1^3\bar{v}_3 + v_3\bar{v}_1^3)]\,,$$
$$[\tilde{u}_1\tilde{u}_3 v_3\bar{v}_3]\,, \quad [\tilde{u}_1\tilde{u}_3 v_1v_2\bar{v}_1^3]\,, \quad [\tilde{u}_1\tilde{u}_3 v_1v_2\bar{v}_1\bar{v}_2]\,.$$

5) *The cokernel is generated by the following classes*, *namely*,

i) *the classes of degree not equal to* 4, 13, 17, 18,

ii) *the class* $[\tilde{u}_2v_1v_2\bar{v}_2] + [\tilde{u}_2v_2\bar{v}_3]$ (*of degree* 13),

iii) *the classes* $[\tilde{u}_1\tilde{u}_3(v_1^3\bar{v}_3 - v_3\bar{v}_1^3)]$, $[\tilde{u}_1\tilde{u}_3(v_1v_2\bar{v}_3 - v_3\bar{v}_1\bar{v}_2)]$, $[\tilde{u}_2\tilde{u}_3v_1v_2\bar{v}_2]$, $[\tilde{u}_2\tilde{u}_3v_2\bar{v}_3]$ *and* $[\tilde{u}_2\tilde{u}_3v_3\bar{v}_2]$ (*of degree* 18), *and*

iv) *the class* $[\bar{v}_1]^2 + 2[\bar{v}_2]$ (*of degree* 4).

REMARK 1.10. Some classes have several different representations, for example,

$$[\tilde{u}_1v_1v_2\bar{v}_1\bar{v}_2] = [\tilde{u}_2v_2\bar{v}_1^3]\,, \quad [\tilde{u}_1\tilde{u}_3v_1v_2\bar{v}_1\bar{v}_2] = [\tilde{u}_2\tilde{u}_3v_2\bar{v}_1^3]\,,$$
$$[\tilde{u}_2\tilde{u}_3v_1^3 + \tilde{u}_1\tilde{u}_3v_1^2\bar{v}_2 + \tilde{u}_1\tilde{u}_3v_1\bar{v}_1\bar{v}_2 - \tilde{u}_1\tilde{u}_2v_1^2\bar{v}_3] = [\tilde{u}_1\tilde{u}_3v_1\bar{v}_1\bar{v}_2 + \tilde{u}_2\tilde{u}_3v_1^2\bar{v}_1]\,.$$

In particular,

$$[u_1v_{J_1}v_{J_2}\bar{v}_{K_1}\bar{v}_{K_2}] = [u_1v_{J_1}v_{K_2}\bar{v}_{K_1}\bar{v}_{J_2}]$$

holds in $H^{4q+1}(\mathrm{WU}_q)$ if $|J_1| + |J_2| = |K_1| + |K_2| = q$ and $|J_1| = |K_1|$.

REMARK 1.11. One might notice that $H^{2q+1}(\mathrm{WU}_q) \cong H^{2q+1}(\mathrm{W}_q) \otimes \boldsymbol{C}$ as vector spaces if $q = 2$ or $q = 3$, where $\mathrm{W}_q = \boldsymbol{R}_q[c_1, \dots, c_q] \otimes \bigwedge[h_1, h_2, \dots, h_q]$ and $H^*(\mathrm{W}_q)$ is the space of real secondary classes of real foliations with trivialized normal bundle. Recently, this turns out to be true in general [4].

In what follows, we always assume that $q > 1$ even though most of the claims remain valid even if $q = 1$.

2. Computational lemmas. First, we examine some relations among the Pontrjagin classes. We denote by C_j and by V_j the j-th Pontrjagin character and the j-th Chern character, respectively. Namely, C_j (resp. V_j) are the Newton polynomials (cf. [10]) evaluated by c_j (resp. v_j). We denote by C_J the monomial $C_1^{j_1} \cdots C_{2q}^{j_{2q}}$, where $J = (j_1, \dots, j_{2q})$. It is easy to verify that $\lambda(C_i) = (\sqrt{-1})^i(V_i + (-1)^i\bar{V}_i)$.

The following lemma is easily shown by using the equation $\lambda(C_i) = (\sqrt{-1})^i(V_i + (-1)^i\bar{V}_i)$.

LEMMA 2.1. *Let* $\mathcal{N}_q$ *denote the kernel of* λ *restricted to* $\boldsymbol{R}_{2q}[c_1, \dots, c_{2q}]$. *Then* $\mathcal{N}_q$ *is generated by the monomials* C_J *such that there are no decompositions* J *into* J_1 *and* J_2 *with* $|J_1| \le q$ *and* $|J_2| \le q$. *Denote by* $\mathcal{K}_q$ *the ideal of* WO_{2q} *generated by* $\mathcal{N}_q$, *namely*, $\mathcal{K}_q = \{h_I\omega \mid \omega \in \mathcal{N}_q\}$. *Then* $\mathcal{K}_q$ *is contained in the kernel of* λ. *In particular*, $\lambda(C_i) = 0$ *if* $i > q$.

For example, C_2^q belongs to $\mathcal{N}_q$ if q is odd. There are also elements of $\ker\lambda$ that do not necessarily belong to $\mathcal{K}_q$.

LEMMA 2.2. *Suppose that $h_I c_J$ is a Vey cocycle and let m be the number of entries of I. Suppose also that $i_{m-t+1}+\cdots+i_m+|J|>(2+t)q-t(t-1)/2$ holds some t $(1\leq t\leq m)$. Then $\lambda(h_I c_J)=0$ in WU_q In particular, we have the following*:

1) $\lambda(h_I c_J)=0$ *in* WU_q *if* $i_m+|J|>3q$.
2) $\lambda(h_I h_{2q-1} c_J)=0$ *in* WU_q *if* $I\neq\phi$.

PROOF. We set $I'=\{i_{m-t+1},\dots,i_m\}$. Then $\lambda(h_{I'}c_J)$ is a linear combination of $\tilde{u}_L v_A\bar{v}_B$ with $|A|+|B|=i_{m-t+1}+\cdots+i_m+|J|-(q+\cdots+(q-t+1))>2q$. Thus $\lambda(h_{I'}c_J)=0$. By setting $t=1$, we obtain 1). By setting $t=2$ and noticing that $i_{m-1}+|J|>2q$, we obtain 2). □

Certain elements with simple index sets belong to $\ker[\lambda]$.

LEMMA 2.3. *Let a be a positive integer and assume that $q+a$ is odd. Then we have the following.*

1) *If $[(|J|+1)/2]+a>q$, then $[h_{q+a}c_J]_\lambda$ is trivial, where $[(|J|+1)/2]$ denotes the largest integer which is not greater than $(|J|+1)/2$.*
2) *If $j+a>q$, then $[h_{q+a}C_j]_\lambda$ is trivial.*

PROOF. Set $l=q+a$. Then, up to multiplication of a constant,

$$h_l=\tilde{u}_q(v_a+\bar{v}_a)-\tilde{u}_{q-1}(v_{a+1}+\bar{v}_{a+1})+\cdots-\tilde{u}_a(v_q+\bar{v}_q)\,,$$

because $q+a$ is odd. We now write $\lambda(c_J)=\sum a_{A,B}(v_A\bar{v}_B\pm v_B\bar{v}_A)$, where $A+B=J$, and assume that $|A|\geq|B|$. It follows from the assumption that $(v_t+\bar{v}_t)(v_A\bar{v}_B\pm v_B\bar{v}_A)=-(v_t-\bar{v}_t)(\pm v_A\bar{v}_B-v_B\bar{v}_A)$, where $a\leq t\leq q$. Hence $\tilde{u}_{q-t}(v_{a+t}+\bar{v}_{a+t})(v_A\bar{v}_B\pm v_B\bar{v}_A)-\tilde{u}_{a+t}(v_{q-t}+\bar{v}_{q-t})(v_A\bar{v}_B\pm v_B\bar{v}_A)$ is an exact form. Therefore $\lambda(h_{q+a}c_J)$ is exact. The second claim can be shown by using the equation $C_j=(\sqrt{-1})^i(V_i+(-1)^i\bar{V}_i)$ and following almost the same argument as above. □

COROLLARY 2.4. *If $|J|>2$, then $[h_{2q-1}c_J]_\lambda=0$. On the other hand, $2[h_{2q-1}c_2]_\lambda=[h_1c_1c_{2q-1}]_\lambda$ holds as elements of $H^{4q+1}(\mathrm{WU}_q)$.*

PROOF. The first equality follows from 1) of Lemma 2.3. The second equality follows from the equation $[h_{2q-1}C_2]_\lambda=0$, which holds by 2) of Lemma 2.3 as we assumed that $q>1$. □

For example, in the case where $q=3$, $[h_5c_4]_\lambda=0$. This cannot be deduced from 1) of Lemma 2.2.

LEMMA 2.5. *If q is odd, then $[h_qc_{2q-1}]_\lambda=[h_qc_{q-1}c_q]_\lambda=0$.*

PROOF. As $q>2$, the following equation holds, namely,

$$\lambda(h_qc_{2q-1})=d\left(\tilde{u}_{q-1}\tilde{u}_q(v_q+\bar{v}_q)-\frac{1}{2}\tilde{u}_1\tilde{u}_{q-1}(v_{q-1}\bar{v}_q+v_q\bar{v}_{q-1})\right),$$

from which the triviality of $[h_qc_{2q-1}]_\lambda$ follows.

We show that $\lambda(h_q c_{q-1} c_q)$ is exact. First, $c_{q-1}c_q$ is a linear combination of the cocycles of the form

$$\begin{aligned} S_{i,j} &= (v_i\bar v_{q-1-i} + v_{q-1-i}\bar v_i)(v_j\bar v_{q-j} - v_{q-j}\bar v_j) \\ &= (v_i v_j \bar v_{q-1-i}\bar v_{q-j} - v_{q-1-i}v_{q-j}\bar v_i\bar v_j) + (v_{q-1-i}v_j\bar v_i\bar v_{q-j} - v_i v_{q-j}\bar v_{q-1-i}\bar v_j)\,, \end{aligned}$$

where $0 \le i, j \le (q-1)/2$. The first term does not vanish if and only if $i + j \le q$ and $2q - (i + j + 1) \le q$. Under the condition we assumed on i and j, it is equivalent to $j = q - i - 1$. The second term does not vanish if and only if $q - 1 - i + j \le q$ and $q + i - j \le q$. These conditions are equivalent to $j = i + 1$ or $i = j$. Thus $c_{q-1}c_q$ is a linear combination of the cocycles of the form

$$\begin{aligned} &v_i v_{q-i-1}\bar v_i\bar v_{q-i} - v_i v_{q-i}\bar v_i\bar v_{q-i-1}\,, \quad \text{and} \\ &v_i v_{q-i-1}\bar v_{i+1}\bar v_{q-i-1} - v_{i+1}v_{q-i-1}\bar v_i\bar v_{q-i-1}\,. \end{aligned}$$

Hence $h_q c_{q-1} c_q$ is a linear combination of the cocycles A_i, B_i, A_i' and B_i' defined respectively by the formulae

$$\begin{aligned} A_i &= \tilde u_q(v_i v_{q-i-1}\bar v_i\bar v_{q-i} - v_i v_{q-i}\bar v_i\bar v_{q-i-1})\,, \\ B_i &= \tilde u_{q-1}(v_1 + \bar v_1)(v_i v_{q-i-1}\bar v_i\bar v_{q-i} - v_i v_{q-i}\bar v_i\bar v_{q-i-1})\,, \\ A_i' &= \tilde u_q(v_i v_{q-i-1}\bar v_{i+1}\bar v_{q-i-1} - v_{i+1}v_{q-i-1}\bar v_i\bar v_{q-i-1})\,, \\ B_i' &= \tilde u_{q-1}(v_1 + \bar v_1)(v_i v_{q-i-1}\bar v_{i+1}\bar v_{q-i-1} - v_{i+1}v_{q-i-1}\bar v_i\bar v_{q-i-1})\,, \end{aligned}$$

where $0 \le i \le (q-1)/2$. First, we assume that $i \ne 0$. Then A_i is exact because in this case

$$\begin{aligned} A_i &= \tilde u_q(v_i - \bar v_i)(v_{q-i-1}\bar v_i\bar v_{q-i} + v_i v_{q-i}\bar v_{q-i-1}) \\ &= -d(\tilde u_q\tilde u_i(v_{q-i-1}\bar v_i\bar v_{q-i} + v_i v_{q-i}\bar v_{q-i-1}))\,. \end{aligned}$$

Here we used the fact that $q - i - 1 \ge (q-1)/2 > 0$ because $q > 1$.

On the other hand, we have the following equation because $2q - 1 > q$, namely,

$$A_0 = \tilde u_q(v_{q-1} - \bar v_{q-1})(v_q + \bar v_q) = -d(\tilde u_q\tilde u_{q-1}(v_q + \bar v_q))\,.$$

Thus A_i is always exact. Finally, as $2q - 2 > q$, the following equation holds, namely,

$$B_i = d(\tilde u_1\tilde u_{q-1}(v_i v_{q-i-1}\bar v_i\bar v_{q-i} + v_i v_{q-i}\bar v_i\bar v_{q-i-1}))\,.$$

Similarly, A_i' and B_i' are also exact. □

Since the image of $H^*(\mathrm{WO}_{2q})$ under $[\lambda]$ is written in terms of $\tilde u_i$, v_j and $\bar v_j$, it consists of the elements of degree at most $q^2 + 4q$. But a slightly careful observation shows that the image is much smaller.

PROPOSITION 2.6. *The image of* $H^*(\mathrm{WO}_{2q})$ *under* $[\lambda]$ *consists of the elements of degree at most* $q^2 - [q/2]^2 + 4q$*, where* $[q/2]$ *denotes the largest integer not greater than* $q/2$*.*

PROOF. We show that the mapping λ annihilates Vey cocycles of degree greater than $q^2 - [q/2]^2 + 4q$. Suppose that $h_I c_J$ is a Vey cocycle such that $\lambda(h_I c_J)$ is non-trivial in WU_q. Let $2i - 1$ be the smallest entry of I. Then $|J| \ge 2q - 2i + 2$, because $h_I c_J$ is a Vey cocycle. It follows from 1) of Lemma 2.2 that $a \le q + 2i - 2$ if $a \in I$. Hence the number of the entries of I is at most $[(q+1)/2]$. Therefore, if we write $\lambda(h_I c_J)$ as a linear combination of the elements of the form $\tilde{u}_I v_J \bar{v}_K$, the degree of the $\tilde{u}_I$-part is at most

$$\sum_{t=q-[(q+1)/2]+1}^{q} (2t-1) = q^2 - [q/2]^2 .$$

On the other hand, the degree of $v_J \bar{v}_K$ is at most $4q$. Consequently, the degree of the cocycle $h_I c_J$ is at most $q^2 - [q/2]^2 + 4q$. □

Finally, we recall the following proposition.

PROPOSITION 2.7 [2, Proposition 3.9]. *Suppose that $|J| > q$ and $|J|$ is even. Then there is a well-defined element η_J of* WU_q *such that $d\eta_J = \lambda(c_J)$.*

3. Writing the image by h_i and c_j with $i, j \le q$. In this section, we study the following question:

(Q) Is it possible to write Im $[\lambda]$ only in terms of h_i and c_j with $i, j \le q$?

Theorem B of [2] shows that this is true if $q = 1$. The following proposition shows that the answer is yes if $q \le 3$.

PROPOSITION 3.1. *The image of $H^*(\mathrm{WO}_{2q})$ under $[\lambda]$ can be written in terms of $h_1, \dots, h_{2q-3}$ and $c_1, \dots, c_q$. In particular, $2[h_{2q-1}c_2]_\lambda = [h_1 c_1 c_{2q-1}]_\lambda$.*

PROOF We can exclude h_{2q-1} by virtue of 2) of Lemma 2.2 and Corollary 2.4. It remains to show the following:

LEMMA 3.2. *Let $h_I c_J$ be a Vey cocycle in* WO_{2q}. *Then the class $[h_I c_J]_\lambda$ is represented by a linear combination of cocycles of the form $[h_{I'} c_{J'}]_\lambda$, where $I' \le I$ and J' admits a decomposition $J' = J_1' + J_2'$ with $|J_1'| \le q$ and $|J_2'| \le q$.*

PROOF. Suppose that J does not admit any decompositions as in the statement. Then $\lambda(h_I C_J) = 0$ by Lemma 2.1, where $C_J = C_1^{j_1} \cdots C_{2q}^{j_{2q}}$. This means that $\lambda(h_I c_J)$ can be written as a linear combination of the cocycles of the form $\lambda(h_I c_{J'})$ with $J' < J$ and $|J'| = |J|$. Therefore, if $I = \{1\}$, then we can deduce the conclusion by the induction on the order of J. If $I \ne \{1\}$, the cocycles $h_I c_{J'}$ as above need not be a Vey cocycle. However, such $h_I c_{J'}$ is cohomologous to a linear combination of Vey cocycles of the form $h_{I'} c_{J''}$ with $I' < I$. This completes the proof of the lemma and Proposition 3.1. □

The following corollary shows that the answer to (Q) is yes if we restrict ourselves to the residual classes, namely, the classes $[h_I c_J]$ with $|J| = 2q$.

COROLLARY 3.3. *The image of a residual class is non-trivial only if it is a linear combination of the classes of the form $[h_I c_J]_\lambda$ with $I \le q$ and $J \le q$.*

PROOF. First, Lemma 2.2 shows that $\lambda(h_I c_J) = 0$ if $i_\infty > q$. We may assume $J \le q$ by Proposition 3.1. □

We use the following classes $[\tilde{h}_i]$ belonging to $\operatorname{Im}[\lambda]$.

DEFINITION 3.4. We define an element $\tilde{h}_{2i+1}$ of WU_q as follows. First, let P_{2i+1} be the Newton polynomial of degree $(2i+1)$, and write $P_{2i+1} = \sum_{k=0}^{i} a_k x_{2k+1} \sigma_k^{2i+1}$. Here σ_k is a polynomial in x_j's, which are uniquely determined by requiring that if x_j appears in σ_k, then either j is even or $j \geq 2k+1$. We denote again by σ_k^{2i+1} the element of WO_{2q} obtained as the polynomial σ_k^{2i+1} evaluated by $c_1, \ldots, c_{2q}$. We now set

$$\tilde{h}_{2i+1} = \sum_{k=0}^{i} a_k \lambda(h_{2k+1} \sigma_k^{2i+1}),$$

and denote by $[\tilde{h}_{2i+1}]$ the induced class of $H^*(\mathrm{WU}_q)$, where $2i+1 > q$.

REMARK 3.5. 1) By a similar argument as in the proof of Lemma 3.1 of [2], one can show that the classes $[\tilde{h}_{2i+1}]$ can be calculated by using real connections. As we do not need this fact here, we omit the proof.

2) Suppose that the complex normal bundle of the foliation is trivial. Then all h_{2k}'s are well-defined. In such a case we can consider also the cocycles $\tilde{h}_{2k}$, $2k > q$ by using the Newton polynomial. For example, if complex codimension is equal to one, then $[\tilde{h}_2] = [\lambda(h_1 c_1 - 2h_2)] = -2\operatorname{Re}\mathrm{Bott}_1$, where $\mathrm{Bott}_1 = u_1 v_1$ (cf. Definition 1.2) is the Bott class for transversely holomorphic foliations with trivial complex normal bundles. See [2] for the details.

The classes $[\tilde{h}_{2i+1}]$ are non-trivial in general.

EXAMPLE 3.6 [5, 7]. Let $M = \boldsymbol{C}^{q+1} \setminus \{0\}$, and consider a holomorphic vector field

$$X(\lambda_0, \ldots, \lambda_q) = \lambda_0 z_0 \frac{\partial}{\partial z_0} + \lambda_1 z_1 \frac{\partial}{\partial z_0} + \cdots + \lambda_q z_q \frac{\partial}{\partial z_q},$$

where $(z_0, \ldots, z_q)$ is the natural coordinate of $\boldsymbol{C}^{q+1}$. Suppose that all λ_i's are non-zero and none of the numbers λ_i/λ_j is a negative real number. We denote by $\mathcal{F}$ the holomorphic flow given by X. Then $\mathcal{F}$ restricts to a transversely holomorphic (real) flow on the unit sphere S^{2q+1} in $\boldsymbol{C}^{q+1}$. By applying the Baum-Bott theory to Chern characters, we see that

$$\int_{S^{2q+1}} \tilde{h}_{q+1} = 2\operatorname{Im}(\sqrt{-1})^{q+2} \frac{\lambda_0^{q+1} + \lambda_1^{q+1} + \cdots + \lambda_n^{q+1}}{\lambda_0 \lambda_1 \cdots \lambda_q}.$$

Here is another example such that the class $[\tilde{h}_{2q+1}]$ is non-trivial.

EXAMPLE 3.7 [3, 2, 12]. Consider the foliation $\tilde{\mathcal{F}}$ of $\mathrm{SL}(q+1, \boldsymbol{C})$ defined as the left cosets of the subgroup H given by $H = \left\{ \begin{pmatrix} a & * \\ 0 & C \end{pmatrix} ; \ C \in \mathrm{GL}(q, \boldsymbol{C}) \right\}$, namely, $\tilde{\mathcal{F}} = \{gH ; g \in \mathrm{SL}(q+1, \boldsymbol{C})\}$. Let Γ be any cocompact uniform lattice of $\mathrm{SL}(q+1, \boldsymbol{C})/T^q$, where T^q is the maximal torus realized as a subgroup of diagonal matrices. Then the manifold $\Gamma \backslash \mathrm{SL}(q+1, \boldsymbol{C})/T^q$ inherits a foliation, say $\mathcal{F}$. We can show that $[\tilde{h}_{2q+1}](\mathcal{F})$ is a multiple of the Godbillon-Vey class. At least for $q \leq 3$, the multiplier is non-zero.

We now state a condition of writing the image of $[\lambda]$ only in terms of h_i and c_j with $i, j \leq q$. Note that the condition in the proposition is clearly necessary.

PROPOSITION 3.8. *Suppose that for any i, $[h_I c_J]_\lambda \cdot [\tilde{h}_{2i+1}] \in [\lambda](H^*(\mathrm{WO}_{2q}^{(2i-1)}))$, where $I \leq q$ and $J \leq q$. Then the image of $H^*(\mathrm{WO}_{2q})$ is represented by linear combinations of the secondary classes of the form $[h_I c_J]_\lambda$, where $I \leq q$ and $J \leq q$.*

PROOF. By Lemma 3.2, it suffices to show that $[\lambda](H^*(\mathrm{WO}_{2q}^{(l)})) = [\lambda](H^*(\mathrm{WO}_{2q}^{(q)}))$ for $l > q$. We proceed by an induction on l. Let $h_I c_J$ be a Vey cocycle such that $i_\infty = l$ and $J \leq q$. Set $I' = I \setminus \{l\}$, and consider the product $[h_{I'} c_J]_\lambda \cdot [\tilde{h}_l]$. As $[h_{I'} c_J]_\lambda \in [\lambda](H^*(\mathrm{WO}_{2q}^{(l-1)}))$, we may assume that $[h_{I'} c_J]_\lambda \cdot [\tilde{h}_l] \in [\lambda](H^*(\mathrm{WO}_{2q}^{(q)}))$. On the other hand, the cocycle $\lambda(h_{I'} c_J)\tilde{h}_l$ is written as a linear combination of the elements of the form $\lambda(h_{I'} c_J h_{2k+1} \sigma_k^l)$, where $2k+1 = 1, 3, \dots, l$. Hence the class $[h_I c_J]_\lambda$ is a linear combination of the classes of the form $[h_{I'} h_{2k+1} c_J \sigma_k^l]_\lambda$ with $2k+1 < l$. □

4. Proof of the theorems.

PROOF OF THEOREM 1.8. The part 1) is obtained by straightforward calculations. By virtue of Propositions 2.6 and 2.7, we see that the secondary classes of degree greater than 11 vanish and that the only non-trivial Pontrjagin class is $[c_2]_\lambda$. There remain the following classes: $[h_3 c_3]_\lambda$, $[h_1 c_4]_\lambda$, $[h_1 c_1 c_3]_\lambda$, $[h_1 c_2^2]_\lambda$, $[h_1 c_1^2 c_2]_\lambda$, $[h_1 c_1^4]_\lambda$ and $[h_3 c_2]_\lambda$.

First, the triviality of $[h_3 c_3]_\lambda$ follows from Corollary 2.4. The relations among the other classes are deduced from Corollary 2.4 and Lemma 2.1 applied to the cocycles $h_1 c_1 C_3$ and $h_1 C_4$. The rest of the claims follows from the following relations:

$$\begin{aligned} \lambda(h_1 c_2^2) &= \sqrt{-1}\,\tilde{u}_1 (2 v_2 \bar{v}_2 + v_1^2 \bar{v}_1^2)\,, \\ \lambda(h_1 c_1^2 c_2) &= \sqrt{-1}\,\tilde{u}_1 (v_1^2 \bar{v}_2 + v_2 \bar{v}_1^2 + 2 v_1^2 \bar{v}_1^2)\,, \\ \lambda(h_1 c_1^4) &= 6\sqrt{-1}\,\tilde{u}_1 v_1^2 \bar{v}_1^2\,. \end{aligned}$$
□

PROOF OF THEOREM 1.9. The part 1) is shown by some calculation using Theorem 1.7. The Pontrjagin classes other than $[c_2]_\lambda$ vanish by Proposition 2.7, and the secondary classes of degree greater than 20 vanish by Proposition 2.6.

Other classes of $H^*(\mathrm{WO}_6)$ are represented by the following Vey cocycles:

19	$h_5 c_5$
18	$h_1 h_3 c_6$, $h_1 h_3 c_1 c_5$, $h_1 h_3 c_2 c_4$, $h_1 h_3 c_1^2 c_4$, $h_1 h_3 c_3^2$, $h_1 h_3 c_1 c_2 c_3$, $h_1 h_3 c_1^3 c_3$, $h_1 h_3 c_2^3$, $h_1 h_3 c_1^2 c_2^2$, $h_1 h_3 c_1^4 c_2$, $h_1 h_3 c_1^6$
17	$h_5 c_4$, $h_5 c_2^2$, $h_3 c_6$, $h_3 c_2 c_4$, $h_3 c_3^2$, $h_3 c_2^3$
15	$h_3 c_5$, $h_3 c_2 c_3$
13	$h_1 c_6$, $h_1 c_1 c_5$, $h_1 c_2 c_4$, $h_1 c_1^2 c_4$, $h_1 c_3^2$, $h_1 c_1 c_2 c_3$, $h_1 c_1^3 c_3$, $h_1 c_2^3$, $h_1 c_1^2 c_2^2$, $h_1 c_1^4 c_2$, $h_1 c_1^6$, $h_3 c_4$, $h_3 c_2^2$, $h_5 c_2$
4	c_2

It follows from 1) of Lemma 2.2 that the class $[h_5c_5]_\lambda$ is trivial. The classes of degree 15 vanish by Lemma 2.5. There remain the classes of degrees 13, 17 and 18.

Among the classes of degree 17, $[h_5c_4]_\lambda$, $[h_5c_2^2]_\lambda$, $[h_3c_2^3]_\lambda$ and $[h_3c_2c_4]_\lambda$ are trivial. First, the triviality of the classes $[h_5c_4]_\lambda$ and $[h_5c_2^2]_\lambda$ follows from Corollary 2.4. The triviality of the class $[h_3c_2^3]_\lambda$ follows from Lemma 2.1 applied to $\lambda(h_3C_2^3)$. Lemma 2.1 also shows that $\lambda(h_3c_2C_4) = 0$. By rewriting this equation in terms of the Vey basis, we obtain the relation $[h_3c_2c_4]_\lambda = (1/2)[h_3c_2^3]_\lambda = 0$. Second, we show that $[h_3c_3^2]_\lambda = 2[h_3c_6]_\lambda \neq 0$ in $H^{17}(\mathrm{WU}_3)$. By rewriting the equation $\lambda(h_3C_6) = 0$ obtained by Lemma 2.1 and using the fact that $[h_3c_2^3]_\lambda$ and $[h_3c_2c_4]_\lambda$ are trivial, we obtain the first equality. The non-triviality follows from the part 1) because $[h_3c_3^2]_\lambda = -2[\sqrt{-1}\tilde{u}_3v_3\bar{v}_3]$.

We now examine the secondary classes of degree 13. The first five relations in the statement are deduced from Lemma 2.1 applied to the cocycles $h_1c_1^2C_4$, $h_1c_2C_4$, $h_1c_1C_5$, h_1C_6 and $h_1C_2^3$. Note that these relations already hold in WU_3.

Next, Corollary 2.4 shows that $2[h_5c_2]_\lambda = [h_1c_1c_5]_\lambda$ in $H^*(\mathrm{WU}_3)$. On the other hand, $\lambda(h_3C_4) = 0$ by Lemma 2.1. From this we see that

$$[h_3c_4]_\lambda = \frac{1}{4}[h_1c_1^3c_3]_\lambda - [h_1c_1c_2c_3]_\lambda + [h_1c_3^2]_\lambda + \frac{1}{2}[h_3c_2^2]_\lambda\,.$$

We now have the following equations:

$$\begin{aligned}
\lambda(h_1c_3^2) &= 2\sqrt{-1}\tilde{u}_1(v_3\bar{v}_3 + v_1v_2\bar{v}_1\bar{v}_2)\,,\\
\lambda(h_1c_1c_2c_3) &= \sqrt{-1}\tilde{u}_1(v_1v_2\bar{v}_3 + v_3\bar{v}_1\bar{v}_2 + v_1^3\bar{v}_1\bar{v}_2 + v_1v_2\bar{v}_1^3 + 2v_1v_2\bar{v}_1\bar{v}_2)\,,\\
\lambda(h_1c_1^3c_3) &= \sqrt{-1}\tilde{u}_1(v_1^3\bar{v}_3 + v_3\bar{v}_1^3 + 3(v_1^3\bar{v}_1\bar{v}_2 + v_1v_2\bar{v}_1^3))\,,\\
\lambda(h_1c_1^2c_2^2) &= \sqrt{-1}\tilde{u}_1(2(v_1^3\bar{v}_1\bar{v}_2 + v_1v_2\bar{v}_1^3) + 2v_1^3\bar{v}_1^3 + 4v_1v_2\bar{v}_1\bar{v}_2)\,,\\
\lambda(h_1c_1^4c_2) &= \sqrt{-1}\tilde{u}_1(4(v_1v_2\bar{v}_1^3 + v_1^3\bar{v}_1\bar{v}_2) + 6v_1^3\bar{v}_1^3)\,,\\
\lambda(h_1c_1^6) &= 20\sqrt{-1}\tilde{u}_1v_1^3\bar{v}_1^3\,,\\
\lambda(h_3c_2^2) &= -\sqrt{-1}\tilde{u}_3(v_1^2\bar{v}_1^2 - 2(v_1v_2\bar{v}_1 + v_1\bar{v}_1\bar{v}_2) + 2v_2\bar{v}_2)\\
&\quad + \frac{\sqrt{-1}}{2}\tilde{u}_2(v_1^3\bar{v}_1^2 + v_1^2\bar{v}_1^3 - 2(v_1v_2\bar{v}_1^2 + v_1^2\bar{v}_1\bar{v}_2))\\
&\quad + \sqrt{-1}\tilde{u}_2(v_2\bar{v}_1\bar{v}_2 + v_1v_2\bar{v}_2)\\
&\quad + 2\sqrt{-1}\tilde{u}_1v_1v_2\bar{v}_1\bar{v}_2\,.
\end{aligned}$$

From the part 1) it follows that these cocycles are linearly independent even if we pass into the cohomology.

Finally, we deal with the secondary classes of degree 18. The relations between the classes of degree 18 follows as in the case of degree 13 by considering h_1h_3 instead of h_1.

On the other hand, the following equations hold in $H^{18}(\mathrm{WU}_3)$:

$$
\begin{aligned}
[h_1h_3c_3^2]_\lambda &= 2[\tilde{u}_1\tilde{u}_3(v_3\bar{v}_3 + v_1v_2\bar{v}_1\bar{v}_2)]\,,\\
[h_1h_3c_1c_2c_3]_\lambda &= [\tilde{u}_1\tilde{u}_3(v_1v_2\bar{v}_3 + v_3\bar{v}_1\bar{v}_2 + v_1^3\bar{v}_1\bar{v}_2 + v_1v_2\bar{v}_1^3 + 2v_1v_2\bar{v}_1\bar{v}_2)]\,,\\
[h_1h_3c_1^3c_3]_\lambda &= [\tilde{u}_1\tilde{u}_3(v_1^3\bar{v}_3 + v_3\bar{v}_1^3 + 3(v_1^3\bar{v}_1\bar{v}_2 + v_1v_2\bar{v}_1^3))]\,,\\
[h_1h_3c_1^2c_2^2]_\lambda &= [\tilde{u}_1\tilde{u}_3(2(v_1^3\bar{v}_1\bar{v}_2 + v_1v_2\bar{v}_1^3) + 2v_1^3\bar{v}_1^3 + 4v_1v_2\bar{v}_1\bar{v}_2)]\,,\\
[h_1h_3c_1^4c_2]_\lambda &= [\tilde{u}_1\tilde{u}_3(4(v_1v_2\bar{v}_1^3 + v_1^3\bar{v}_1\bar{v}_2) + 6v_1^3\bar{v}_1^3)]\,,\\
[h_1h_3c_1^6]_\lambda &= 20[\tilde{u}_1\tilde{u}_3v_1^3\bar{v}_1^3]\,.
\end{aligned}
$$

Now the linear independence of these classes follows again from the part 1). The proof is completed. □

REFERENCES

[1] T. ASUKE, On the real secondary classes of transversely holomorphic foliations, University of Tokyo, Thesis.

[2] T. ASUKE, On the real secondary classes of transversely holomorphic foliations, Ann. Inst. Fourier 50 (2000), 995–1017.

[3] T. ASUKE, The Godbillon-Vey class of transversely holomorphic foliations, preprint.

[4] T. ASUKE, Complexification of foliations and complex secondary classes, to appear in Bull. Braz. Math. Soc. (N.S.).

[5] P. BAUM AND R. BOTT, Singularities of Holomorphic Foliations, J. Differential Geom. 7 (1972), 279–342.

[6] R. BOTT, On the Lefschetz Formula and Exotic Characteristic Classes, Symposia Math. 10 (1972), 95–105.

[7] R. BOTT, S. GITLER AND I. M. JAMES, Lectures on Algebraic and Differential Topology, Lecture Notes in Math. 279, Springer-Verlag, Berlin, 1972.

[8] R. BOTT AND A. HAEFLIGER, On characteristic classes of Γ-foliations, Bull. Amer. Math. Soc. 78 (1972), 1039–1044.

[9] C. GODBILLON, Cohomologies d'algèbres de Lie de champs de vecteurs formels, Séminaire Bourbaki (1972/73), No. 421, Lecture Notes in Math. 383, Springer-Verlag, Berlin, 1974, pp. 69–87.

[10] D. HUSEMOLLER, Fibre Bundles, Graduate Texts in Mathematics 20, Springer-Verlag, New York, 1994.

[11] H. V. PITTIE, Characteristic classes of foliations, Research Notes in Mathematics 10, Pitman Publishing, London-San Francisco, Calif.-Melbourne, 1976.

[12] O. H. RASMUSSEN, Exotic Characteristic Classes for Holomorphic Foliations, Invent. Math. 46 (1978), 153–171.

DEPARTMENT OF MATHEMATICS
(YOSHIDA-MINAMI BRANCH)
KYOTO UNIVERSITY
KYOTO 606–8501
JAPAN

E-mail address: asuke@math.kyoto-u.ac.jp