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HUYGENS OPERATORS ON PRODUCT MANIFOLDS

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Abstract. Based on two equalities for power series which are equivalent to the Tedone formulas, the elementary solution to the wave operator on the product of k Riemannian manifolds is represented as a composition, with respect to the time variable, of k elementary solutions to wave operators on factor manifolds. As a consequence, one has an infinite number of non-trivial momentary Huygens operators. For example, wave operators on the product of an odd numer of odd dimensional manifolds with constant curvature are revealed to be momentary Huygens operators for an appropriate choice of coefficients of the 0-th order terms.

1. Main results. Let (M, g) be an oriented, compact or non-compact complete Riemannian manifold of class C^∞ *without boundary* and t be a time variable. We denote the Laplace-Beltrami operator on (M, g) by Δ_M. In this article, we suppose that the metric g be independent of t. Then the Cauchy problem for generalized wave equation

$$\frac{\partial^2 u}{\partial t^2} - \Delta_M u + cu = 0\,; \tag{1}$$

$$u(p, 0) = 0\,, \quad \frac{\partial u}{\partial t}(p, 0) = f(p)\,, \tag{2}$$

c being a given constant, has a unique solution u for long period $(-\infty < t < +\infty)$ for every function f of class C^∞ on M. Let us denote it in the following way:

$$u(p, t) = \langle G(p, t, \cdot; c), f\rangle = \int_M G(p, t, q; c) f(q) dv(q)\,, \quad p \in M\,, \ \ t \in \boldsymbol{R}\,, \tag{3}$$

where dv is the volume element of (M, g). $G(p, t, \cdot; c)$ is a function of class C^∞ on $M \times \boldsymbol{R}$ with values in distributions on M, and is an odd function of t. G is said to be the *elementary solution* to the Cauchy problem for generalized wave operator

$$P_c = \frac{\partial^2}{\partial t^2} - \Delta_M + c\,, \quad \text{where } c \text{ is a constant}\,.$$

Let r be the geodesic distance between two adjacent points of M. Then, $G(p, t, \cdot; c)$ has support in the closed geodesic ball $\{q \in M;\ r(p, q) \le |t|\}$ if $|t|$ is sufficiently small. We say that G has *momentarily a strong lacuna*, or that P_c is a *momentary Huygens operator*, if, for every point p of M, there exists a positive number $T(p)$ such that $G(p, t, \cdot; c)$ has support in the geodesic sphere $\{q \in M;\ r(p, q) = |t|\}$ provided that $|t| < T(p)$. If P_c is a momentary Huygens operator, the dimension of M is odd and not smaller than 3. This is one of theorems established by Hadamard [H]. The elementary solution for long period is obtained by iteration

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of that for short period. However, to discuss the strong lacuna for long period, we need some global considerations in geometry. So, in the present article, we concentrate our study only to the question for short period.

We will prove the following theorem in Section 4.

THEOREM 1. *Given a Riemannian manifold* (M, g), *there exists at most one constant* c *such that* P_c *be a momentary Huygens operator. Furthermore, such* c *is a real number if exists.*

Next, let $(M^{(j)}, g^{(j)})$, $j = 1, 2, \dots, k$, be k Riemannian manifolds. Then the product space $M^{(1)} \times \cdots \times M^{(k)}$ is also a Riemannian manifold endowed with the metric $g^{(1)} + \cdots + g^{(k)}$. We denote a point of the product space by $p = (p^{(1)}, \dots, p^{(k)})$, $p^{(j)} \in M^{(j)}$. If we denote by r_j the geodesic distance between two adjacent points in $M^{(j)}$ and by r the geodesic distance between two adjacent points in the product space, we have the Pythagoras formula:

$$r(p, q)^2 = \sum_{j=1}^{k} r_j(p^{(j)}, q^{(j)})^2 . \tag{4}$$

The following is the main result of this article. We shall prove it in Section 2.

THEOREM 2. *Given* k *Riemannian manifolds* $(M^{(j)}, g^{(j)})$, *let* $G^{(j)}(p^{(j)}, t, q^{(j)}; c^{(j)})$ *be the elementary solution to the Cauchy problem for*

$$P_j = \frac{\partial^2}{\partial t^2} - \Delta_{M^{(j)}} + c^{(j)}$$

on $(M^{(j)}, g^{(j)})$, $1 \le j \le k$. *Suppose that, for every point* $p^{(j)}$ of $M^{(j)}$, *there exists a positive number* $T(p^{(j)})$ *such that the first order derivative* $\partial G^{(j)}(p^{(j)}, t, \cdot; c^{(j)})/\partial t$ *has support in the sphere* $\{q^{(j)} \in M^{(j)}; r_j(p^{(j)}, q^{(j)}) = |t|\}$ *provided that* $|t| < T(p^{(j)})$ *for every* j, *and that* k *be odd. Then the operator*

$$Q = \frac{\partial^2}{\partial t^2} - \sum_{j=1}^{k} \Delta_{M^{(j)}} + \sum_{j=1}^{k} c^{(j)}$$

on the product manifold $(M^{(1)} \times \cdots \times M^{(k)}, g^{(1)} + \cdots + g^{(k)})$ *is a momentary Huygens operator. In particular, if* $P_1, \dots, P_k$ *are momentary Huygens operators and if* k *is odd,* Q *is also a momentary Huygens operator.*

The Huygens property has been systematically investigated by Günther [G_2] for operators with variable coefficients. To be precise, let P be a second order hyperbolic operator of metric principal part in a ν-dimensional curved space-time N endowed with a metric h of signature $(+, -, \cdots, -)$ $(\nu \ge 2)$. Here, P is said to be of *metric principal part* if the second order part of P coincides with that of the Laplace-Beltrami operator with respect to h. Given a real-valued smooth function φ and a positive smooth function a in a neighborhood of a point in N, we can define a conformal change $h \mapsto e^{2\varphi} h$ of metric and a gauge transformation

$u \mapsto au$ of unknown function on N. So, we have a new operator $\tilde{P}$ by setting

$$\tilde{P}u = e^{-(\nu+2)\varphi/2} P[e^{(\nu-2)\varphi/2} au]/a\,. \tag{5}$$

$\tilde{P}$ is said to be the *conformal gauge transform* of P depending on (φ, a) (see Cotton [C]). Following Günther [G_1], we say a hyperbolic operator P of metric principal part to be *trivial* if every point z of N has a neighborhood V such that an appropriate conformal gauge transform $\tilde{P}$ of P in V has an expression

$$\tilde{P} = \frac{\partial^2}{\partial {y_1}^2} - \frac{\partial^2}{\partial {y_2}^2} - \cdots - \frac{\partial^2}{\partial {y_\nu}^2}$$

in an appropriate local coordinate system $(y_1, y_2, \ldots, y_\nu)$. The operator on the right hand side is said to be the *d'Alembertian* of ν independent variables.

An operator is a momentary Huygens operator if one of its conformal gauge transforms is. A trivial operator is a momentary Huygens operator if $\nu = \dim N$ is even and $\nu \geq 4$. Nishiwada [N] has proposed an important class of non-trivial Huygens operators.

We have the following corollary to Theorem 2 which will be proved in Section 3.

COROLLARY. *There exists an infinite number of non-trivial momentary Huygens operators.*

The author expresses his sincere gratitude to Professor Daisuke Fujiwara who informed him of the preprint of Kannai [K] after the preparation of this work.

2. Proof of Theorem 2. The existence and uniqueness of the solution to the Cauchy problem (1)–(2) allows us to represent the solution in the following way.

$$u(\cdot, t) = \frac{\sin(t\sqrt{A})}{\sqrt{A}} f\,, \quad \frac{\partial u}{\partial t}(\cdot, t) = \cos(t\sqrt{A}) f \quad \text{where} \quad A = -\Delta_M + c\,. \tag{6}$$

LEMMA 1. *Let $G(p, t, q; \sum_{j=1}^{k} c^{(j)})$ be the elementary solution to Q on the product manifold defined in Theorem 2. Then the following hold.*

(i) *If k is odd and $k \geq 3$, we set $k = 2h + 3$. Then we have*

$$\begin{aligned} &G\left(p, t, q; \sum_{j=1}^{k} c^{(j)}\right) \\ &\quad = \alpha_k \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^h \left\{ t^{k-2} \int \cdots \int_{S^{k-1}} \prod_{j=1}^{k} \frac{\partial G^{(j)}}{\partial t}(p^{(j)}, \omega_j t, q^{(j)}; c^{(j)}) dS_\omega \right\}, \end{aligned} \tag{7}$$

where

$$1/\alpha_k = 2^{(k+1)/2} \pi^{(k-1)/2}\,,$$

$\omega = (\omega_1, \ldots, \omega_k)$ is the generic point of the sphere S^{k-1} of radius 1 centered at the origin of $\boldsymbol{R}^k$ and dS is the surface element of S^{k-1} induced from the Euclidean metric of $\boldsymbol{R}^k$.

(ii) *If k is even and $k \geq 2$, we set $k = 2h + 2$. Then we have*

$$G\Big(p, t, q; \sum_{j=1}^{k} c^{(j)}\Big) \tag{8}$$
$$= \beta_k \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^h \left\{ t^{k-1} \int \cdots \int_{B^k} \prod_{j=1}^{k} \frac{\partial G^{(j)}}{\partial t}(p^{(j)}, u_j t, q^{(j)}; c^{(j)}) \frac{du}{\sqrt{1-|u|^2}} \right\},$$

where

$$1/\beta_k = (2\pi)^{k/2},$$

B^k is the ball of radius 1 centered at the origin of $\boldsymbol{R}^k$ and $du = du_1 \cdots du_k$.

PROOF OF LEMMA 1. Set $A_j = -\Delta_{M^{(j)}} + c^{(j)}$ for $1 \leq j \leq k$. Then we have k operators $A_1, \dots, A_k$ which commute each other. Given a smooth function f on $M = M^{(1)} \times \cdots \times M^{(k)}$, let u_f be the solution to the Cauchy problem (1)–(2) for Q. Analogously to (6), we have

$$u_f(\cdot, t) = \frac{\sin(t\sqrt{A_1 + \cdots + A_k})}{\sqrt{A_1 + \cdots + A_k}} f. \tag{9}$$

We shall prove in Section 6.1 the following two equalities involving arbitrary complex numbers $a_1, \dots, a_k$, where h, α_k, β_k are the same as in (7), (8), respectively. If k is odd and $k \geq 3$, then

$$\frac{\sin(t\sqrt{a_1^2 + \cdots + a_k^2})}{\sqrt{a_1^2 + \cdots + a_k^2}} = \alpha_k \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^h \left\{ t^{k-2} \int \cdots \int_{S^{k-1}} \prod_{j=1}^{k} \cos(a_j \omega_j t) dS_\omega \right\}. \tag{10}$$

If k is even and $k \geq 2$, then

$$\frac{\sin(t\sqrt{a_1^2 + \cdots + a_k^2})}{\sqrt{a_1^2 + \cdots + a_k^2}} = \beta_k \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^h \left\{ t^{k-1} \int \cdots \int_{B^k} \prod_{j=1}^{k} \cos(a_j u_j t) \frac{du}{\sqrt{1-|u|^2}} \right\}. \tag{11}$$

(See also Kannai [K].) (9) and (10) imply, for the odd case,

$$u_f(p, t) = \alpha_k \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^h \left\{ t^{k-2} \int \cdots \int_{S^{k-1}} v_f(p, t, \omega) dS_\omega \right\}, \tag{12}$$

where

$$v_f(\cdot, t, \omega) = \cos(\omega_1 t\sqrt{A_1}) \cdots \cos(\omega_k t\sqrt{A_k}) f.$$

Therefore, we have (7). Analogously, (11) implies (8) for the even case.

On the right hand side of (7), we first calculate $\partial G^{(j)}(p^{(j)}, s, q^{(j)}; c)/\partial s$ and then substitute $s = \omega_j t$. A similar computation is to be done also for (8). q.e.d.

PROOF OF THEOREM 2. The function v_f introduced in (12) has an expression

$$v_f(p, t, \omega) = \int_{M^{(1)}} \cdots \int_{M^{(k)}} f(q) \prod_{j=1}^{k} \frac{\partial G^{(j)}}{\partial t}(p^{(j)}, \omega_j t, q^{(j)}; c^{(j)}) dv_1(q^{(1)}) \cdots dv_k(q^{(k)}).$$

We fix a point p of M and a small positive time t. Suppose that f be smooth and have support in a geodesic ball of radius δ centered at q^0 and that $r(p,q^0)+\delta < t$. Then we have $r(p,q) < t$ for every point q on the support of f. For p,t,q fixed, we define $U_j = \{\omega \in S^{k-1}; |\omega_j|t > r_j(p^{(j)},q^{(j)})\}$, $1 \le j \le k$. From (4), $\{U_j\}_{j=1}^k$ is an open covering of S^{k-1}. So, $v_f(p,t,\omega)$ vanishes identically on S^{k-1} by hypothesis on $\partial G^{(j)}/\partial t$. This implies that $u_f(p,t) = 0$, which is true for every smooth function f with support in the geodesic ball $\{q \in M; r(p,q) < t\}$. Consequently, $G(p,t,\cdot;\sum_{j=1}^k c^{(j)})$ has support in the geodesic sphere $\{q \in M; r(p,q) = t\}$. q.e.d.

REMARK 1. The difference between (10) and (11) is important. To be more precise, *if* $k \ge 2$, *there exists a distribution* Φ *on* $\boldsymbol{R}^k$ *with support in* S^{k-1} *such that*

$$\frac{\sin\sqrt{a_1{}^2+\cdots+a_k{}^2}}{\sqrt{a_1{}^2+\cdots+a_k{}^2}} = \left\langle \Phi, \prod_{j=1}^k \cos(a_j p_j)\right\rangle \tag{13}$$

for arbitrary complex numbers $a_1, \ldots, a_k$ *if and only if* k *is odd.*

We shall prove this in Section 6.5. This explains why the d'Alembertian of $k+1$ variables is or is not a Huygens operator according to the parity of k (see (14), (15) and (16) below).

REMARK 2. (8) or its variant (37) in Section 6.3 may also be useful to discuss the strong lacuna. In Section 6.4, we shall show this for one of the simplest examples.

3. Fundamental examples of trivial operators. Proof of Corollary. In this paper, we confine our investigation to an operator of type

$$P_c = \frac{\partial^2}{\partial t^2} - \Delta_M + c$$

on a Riemannian manifold M endowed with a metric g independent of t. The space-time $M \times \boldsymbol{R}$ has a metric $h = dt^2 - g$, and P_c is of metric principal part. We denote $n = \dim M$ and suppose that $n \ge 3$ for simplicity.

The following is a particular case of a result established by Günther. We prove it in Section 5 for the sake of completeness.

LEMMA 2 ([G$_2$, pp. 486–494]). (i) *Take a small open subset* U *of* M *and a small open interval* I *of* $\boldsymbol{R}$. *If there exists a smooth function* φ *such that* $e^{2\varphi}h$ *be a flat metric in* $U \times I$, *then* g *is of constant sectional curvature in* U.

(ii) *If* g *is of constant sectional curvature* σ *in* $U \times I$, *then* P_c *is trivial in* $U \times I$ *if and only if* $c = \sigma(n-1)^2/4$.

There are three fundamental examples of trivial operators.

(1°) The d'Alembertian of $n+1$ variables

$$\frac{\partial^2}{\partial t^2} - \sum_{j=1}^n \frac{\partial^2}{\partial p_j{}^2}, \quad (p,t) \in \boldsymbol{R}^n \times \boldsymbol{R},$$

is trivial. It is a Huygens operator if and only if n is odd and not smaller than 3. The solution u to the Cauchy problem (1)–(2) is given by classical formulas. The d'Alembert formula for operator representing the vibration of an infinite string ($n = 1$) is given as follows:

$$u(p,t) = \frac{1}{2}\int_{-t}^{t} f(p+q)dq\,, \quad (p,t) \in \boldsymbol{R} \times \boldsymbol{R}\,. \tag{14}$$

We reproduce the Tedone formulas ([T]). If n is odd and $n \geq 3$, we set $n = 2m+3$. Then,

$$u(p,t) = \alpha_n \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^m \left\{ t^{n-2} \int \cdots \int_{S^{n-1}} f(p+t\omega)dS_\omega \right\}, \quad (p,t) \in \boldsymbol{R}^n \times \boldsymbol{R}\,. \tag{15}$$

If n is even and $n \geq 2$, we set $n = 2m+2$. Then,

$$u(p,t) = \beta_n \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^m \left\{ t^{n-1} \int \cdots \int_{|q|<1} \frac{f(p+tq)}{\sqrt{1-|q|^2}}dq \right\}, \quad (p,t) \in \boldsymbol{R}^n \times \boldsymbol{R}\,. \tag{16}$$

Here, α_n, β_n are the same as in (7), (8), respectively.

(15), (16) are proved by means of the Fourier analysis in most of textbooks. Now, we can verify them easily from (7), (8), respectively, and (14) because $\boldsymbol{R}^n = \boldsymbol{R} \times \cdots \times \boldsymbol{R}$.

(2°) The operator

$$P = \frac{\partial^2}{\partial t^2} - \Delta_{S^n} + \left(\frac{n-1}{2}\right)^2$$

on the unit sphere S^n is trivial for any dimension n, and it is a momentary Huygens operator if and only if n is odd and $n \geq 3$. We can define a local coordinate system in every hemisphere×interval in the following way for any n odd or even. Let $\omega = (\omega_0, \omega_1, \dots, \omega_n)$ be the coordinates in $\boldsymbol{R}^{n+1}$ of a point of S^n. We set

$$y_0 = \frac{\sin t}{\psi}\,, \quad y_j = \frac{\omega_j}{\psi}\,, \quad 1 \leq j \leq n\,, \tag{17}$$

where

$$\psi = \psi(\omega, t) = \omega_0 + \cos t\,,$$

for example, in the region $\{(\omega, t) \in S^n \times \boldsymbol{R};\, \omega_0 > 0,\, |t| < \pi/2\}$. Then we have

$$Pu = \psi^{-(n+3)/2}\left(\frac{\partial^2}{\partial {y_0}^2} - \sum_{j=1}^{n} \frac{\partial^2}{\partial {y_j}^2}\right)(\psi^{(n-1)/2}u)\,. \tag{18}$$

A background of (18) will be sketched in Section 5.

The solution to the Cauchy problem (1)–(2) is obtained from (18) together with the Tedone formula (15) or (16). If n is odd and $n \geq 3$, we set $n = 2m+3$. Then,

$$u(p,t) = \alpha_n \left(\frac{1}{\sin t}\frac{\partial}{\partial t}\right)^m \left\{ (\sin t)^{n-2} \int \cdots \int_{\Sigma(p,t)} f(q)d\Sigma_q \right\}, \quad p \in S^n, 0 < t < \pi\,. \tag{19}$$

Here we denote $\Sigma(p,t) = \{q \in S^n;\, r(p,q) = t\}$, which is a sphere of dimension $n-1$ of the Euclidean radius $\sin t$, so we represent every point of $\Sigma(p,t)$ by means of a point of the unit sphere S^{n-1}. $d\Sigma$ stands for the volume element of S^{n-1}. The solution in a long period is obtained if we extend u of (19) to an odd function of t in the interval $(-\pi, \pi)$ and to a periodic function of period 2π for all t. If n is even and $n \geq 2$, we set $n = 2m + 2$. Then,

$$u(p,t) = \beta_n \left(\frac{1}{\sin t}\frac{\partial}{\partial t}\right)^m \int \cdots \int_{\Omega(p,t)} \frac{f(q)}{\sqrt{2\langle p,q\rangle - 2\cos t}} dS_q\,, \quad p \in S^n\,, \quad 0 < t < \pi\,, \tag{20}$$

where $\Omega(p,t) = \{q \in S^n; r(p,q) < t\}$ and dS is the volume element of S^n.

(3°) The operator

$$P = \frac{\partial^2}{\partial t^2} - \Delta_{H^n} - \left(\frac{n-1}{2}\right)^2$$

on the hyperbolic space $\boldsymbol{H}^n$ with sectional curvature -1 is trivial for any dimension n, and it is a Huygens operator if and only if n is odd and $n \geq 3$. We identify $\boldsymbol{H}^n$ with the set $\boldsymbol{R}^n$ whose geodesic distance between two points p, q is defined to be the non-negative number $r(p,q)$ satisfying $\cosh r(p,q) = p_0 q_0 - \langle p, q\rangle$, where $\langle p, q\rangle = \sum_{j=1}^n p_j q_j$ and $p_0 = \sqrt{1 + \langle p, p\rangle}$. In terms of a global coordinate system $(p_1, \ldots, p_n)$ in $\boldsymbol{H}^n$, we can define $(y_0, y_1, \ldots, y_n)$ in the following way for any n odd or even.

$$y_0 = \frac{\sinh t}{\psi}\,, \quad y_j = \frac{p_j}{\psi}\,, \quad 1 \leq j \leq n\,, \tag{21}$$

where

$$\psi = \psi(p,t) = p_0 + \cosh t$$

(see Section 5). Then we have

$$Pu = \psi^{-(n+3)/2}\left(\frac{\partial^2}{\partial y_0^2} - \sum_{j=1}^n \frac{\partial^2}{\partial y_j^2}\right)(\psi^{(n-1)/2}u)\,. \tag{22}$$

The solution to the Cauchy problem (1)–(2) is given by the following formulas. If n is odd and $n \geq 3$, we set $n = 2m + 3$. Then,

$$u(p,t) = \alpha_n \left(\frac{1}{\sinh t}\frac{\partial}{\partial t}\right)^m \left\{(\sinh t)^{n-2} \int \cdots \int_{\Sigma(p,t)} f(q)\, d\Sigma_q\right\}, \quad p \in \boldsymbol{H}^n, t > 0\,, \tag{23}$$

where $\Sigma(p,t) = \{q \in \boldsymbol{H}^n; r(p,q) = t\}$. We represent a point of $\Sigma(p,t)$ by means of a point of S^{n-1}. $d\Sigma$ is the volume element of S^{n-1}. If n is even and $n \geq 2$, we set $n = 2m + 2$. Then,

$$u(p,t) = \beta_n \left(\frac{1}{\sinh t}\frac{\partial}{\partial t}\right)^m \int \cdots \int_{\Omega(p,t)} \frac{f(q)}{\sqrt{2\cosh t - 2\cosh r(p,q)}} dv_q\,, \quad t > 0\,, \tag{24}$$

where $\Omega(p,t) = \{q \in \boldsymbol{H}^n; r(p,q) < t\}$ and $dv_q = (1/q_0)dq_1 \cdots dq_n$.

PROOF OF COROLLARY TO THEOREM 2. Let k be odd and $k \geq 3$. For every j, $1 \leq j \leq k$, let $(M^{(j)}, g^{(j)})$, of *odd* dimension n_j, be either the Euclidean space, sphere or hyperbolic space. Denoting the sectional curvature of $g^{(j)}$ by σ_j, we set $c^{(j)} = \sigma_j(n_j - 1)^2/4$. Then, Q in Theorem 2 is a momentary Huygens operator. Unless all $M^{(j)}$'s are Euclidean, Q is non-trivial because the metric $g^{(1)} + \cdots + g^{(k)}$ on $M^{(1)} \times \cdots \times M^{(k)}$ is not of constant sectional curvature. For a fixed k, various choice of sectional curvature of every factor space gives rise to an infinite number of non-trivial momentary Huygens operators. q.e.d.

4. Proof of Theorem 1.

LEMMA 3. *The following equality holds for arbitrary complex numbers* c, γ:

$$G(p,t,q;c+\gamma) = \int_0^t J_0(\sqrt{\gamma(t^2-s^2)})\frac{\partial}{\partial s}G(p,s,q;c)ds\,, \tag{25}$$

where $J_0(z) = \sum_{m=0}^{\infty}(-z^2)^m/(4^m m!^2)$ *is the Bessel function of order* 0.

PROOF OF LEMMA 3. Power series expansion and term by term integration yield

$$\frac{\sin\sqrt{a^2+b^2}}{\sqrt{a^2+b^2}} = \int_0^1 J_0(b\sqrt{1-s^2})\cos(as)ds\,. \tag{26}$$

(25) follows from this and (6). q.e.d.

Denoting by $u_f(p,t)$ the solution to the Cauchy problem (1)–(2) for

$$P_c = \frac{\partial^2}{\partial t^2} - \Delta_M + c\,,$$

we set

$$v_f(p,t) = \int_0^t J_0(\sqrt{\gamma(t^2-s^2)})\frac{\partial}{\partial s}u_f(p,s)ds\,. \tag{a_1}$$

LEMMA 4. *Given a non-zero complex number* γ, *a positive number* R *and a continuous function* $\phi(s)$ *in* $0 \le s \le R$, *we set*

$$\psi(t) = \int_0^t J_0(\sqrt{\gamma(t^2-s^2)})\phi(s)ds\,, \quad 0 \le t \le R\,. \tag{a_2}$$

Suppose that there exist two more real numbers R_1, R_2 *such that*

$$0 < R_1 \le R_2 < R\,, \quad \phi(s) = 0 \quad \textit{if} \;\; R_1 \le s \le R\,, \quad \psi(t) = 0 \quad \textit{if} \;\; R_2 \le t \le R\,.$$

Then, $\phi(s)$ *is identically equal to zero in the interval* $0 \le s \le R$.

PROOF OF LEMMA 4. By the assumption on the support of ϕ, we have

$$\psi(t) = \int_0^{R_1} J_0(\sqrt{\gamma(t^2-s^2)})\phi(s)ds \quad \text{if} \;\; R_1 \le t \le R\,.$$

The domain of integration is independent of t, so the right hand side is extended to an entire function of t. On the other hand, the left hand side vanishes in a non-empty open interval $R_2 < t < R$ by the assumption on the support of ψ. Therefore, we have

$$\int_0^{R_1} J_0(\sqrt{\gamma(t^2-s^2)})\phi(s)ds = 0$$

identically by the theorem of identity. We operate $(t^{-1}\partial/\partial t)^j$ and multiply by $j!$ to both sides to have

$$\int_0^{R_1} \sum_{m=0}^{\infty} \frac{(-\gamma/4)^m j!}{m!(m+j)!}(t^2 - s^2)^m \phi(s) ds = 0 .$$

Letting $j \to \infty$, we see that the integral of $\phi(s)$ vanishes by the dominated convergence theorem. We eliminate the term with $m = 0$ from the integrand, multiply again by $j + 1$ and tend to the limit as $j \to \infty$. After repeating this procedure, we set $t = 0$. Then, we have successively

$$\int_0^{R_1} s^{2m} \phi(s) ds = 0 , \quad m = 0, 1, 2, \dots .$$

Therefore $\phi(s)$ is orthogonal to every polynomial of s^2 and hence to every continuous function in the interval $0 \le s \le R_1$. So, $\phi(s)$ is identically equal to 0 in the interval $0 \le s \le R_1$ and hence in $0 \le s \le R$. q.e.d.

PROOF OF THEOREM 1. Suppose that both P_c and $P_{c+\gamma}$ be momentary Huygens operators and that $\gamma \ne 0$. Then, for every point p^0 of M, there exists a positive number $T = T(p^0)$ such that the support of $G(p^0, t, \cdot; c)$ and the support of $G(p^0, t, \cdot; c + \gamma)$ are both contained in the geodesic sphere $\{q \in M; r(p^0, q) = |t|\}$ provided that $|t| < T$.

Let f be an arbitrary function of class C^∞ on M non-vanishing at p^0 and with support contained in a geodesic ball $\{q \in M; r(p^0, q) < \delta\}$, where we suppose that $0 < \delta < T$. The equality (a_2) holds for

$$\phi(s) = \partial u_f(p^0, s)/\partial s , \quad \psi(t) = v_f(p^0, t) ,$$

$\phi(s) = 0$ if $\delta \le s \le T$ and $\psi(t) = 0$ if $\delta \le t \le T$ because both P_c and $P_{c+\gamma}$ are supposed to be momentary Huygens operators. Then, we can apply Lemma 4 by setting $R_1 = R_2 = \delta$ and $R = T$ to conclude that $\phi(s)$ is identically equal to zero on $0 \le s \le T$. So, $f(p^0) = \lim_{s \downarrow 0} \phi(s) = 0$ contrarily to our assumption. Hence, at least one of P_c and $P_{c+\gamma}$ is not a momentary Huygens operator.

Next, remark that $G(p, t, q; \bar{c}) = \overline{G(q, t, p; c)}$ for any complex number c because Δ_M is symmetric. So, if P_c is a momentary Huygens operator, then so is $P_{\bar{c}}$. The uniqueness of c implies $\bar{c} = c$ and hence c is a real number. q.e.d

5. Proof of Lemma 2, (18) and (22).

PROOF OF (i). If $(s^1, \dots, s^n)$ is a local coordinate system in U, g has an expression $g_{\alpha\beta} ds^\alpha ds^\beta$ in U. We denote $t = s^0$ and $h = dt^2 - g = h_{jk} ds^j ds^k$. Here and in what follows in this section, Roman indices $j, k, \dots$ range from 0 to n, Greek indices $\alpha, \beta, \dots$ range from 1 to n and we make use of the summation convention of Einstein.

Define the Christoffel symbols and the curvature tensors of g and h to be

$$\Gamma_{\alpha\lambda\beta} = (1/2)(\partial_\alpha g_{\beta\lambda} + \partial_\beta g_{\alpha\lambda} - \partial_\lambda g_{\alpha\beta}) , \quad \bar{\Gamma}_{jrk} = (1/2)(\partial_j h_{kr} + \partial_k h_{jr} - \partial_r h_{jk}) ,$$

$$R_{\alpha\beta\gamma\delta} = \partial_\gamma \Gamma_{\alpha\beta\delta} - \partial_\delta \Gamma_{\alpha\beta\gamma} + g^{\lambda\mu}(\Gamma_{\alpha\lambda\gamma}\Gamma_{\beta\mu\delta} - \Gamma_{\alpha\lambda\delta}\Gamma_{\beta\mu\gamma}) ,$$

$$\bar{R}_{jkrs} = \partial_r \bar{\Gamma}_{jks} - \partial_s \bar{\Gamma}_{jkr} + h^{pq}(\bar{\Gamma}_{jpr}\bar{\Gamma}_{kqs} - \bar{\Gamma}_{jps}\bar{\Gamma}_{kqr}) ,$$

respectively, where $\partial_j = \partial/\partial s^j$. Since $h_{00} = 1$, $h_{0\alpha} = 0$ and $h_{\alpha\beta} = -g_{\alpha\beta}$, we can verify that $\bar{R}_{jkrs} = 0$ if $0 \in \{j, k, r, s\}$ and that $\bar{R}_{\alpha\beta\gamma\delta} = -R_{\alpha\beta\gamma\delta}$. The curvature tensor $\hat{R}_{jkrs}$ of $e^{2\varphi}h$ is obtained if we replace h_{jk} by $e^{2\varphi}h_{jk}$. We have (see Eisenhart [E., p. 89])

$$e^{-2\varphi}\hat{R}_{jkrs} - \bar{R}_{jkrs} = h_{kr}a_{js} + h_{js}a_{kr} - h_{ks}a_{jr} - h_{jr}a_{ks} + (h_{jr}h_{ks} - h_{js}h_{kr})b\,, \tag{27}$$

where

$$a_{jk} = e^{\varphi}\bar{\nabla}_j\bar{\nabla}_k e^{-\varphi} = a_{kj}\,, \quad b = h^{jk}(\partial_j\varphi)(\partial_k\varphi)\,,$$

and $\bar{\nabla}_j = \bar{\nabla}_{\partial/\partial s^j}$ is the covariant differentiation with respect to h. To be more precise,

$$(b_1)\qquad \begin{gathered} a_{00} = e^{\varphi}{\partial_0}^2 e^{-\varphi}\,, \quad a_{0\alpha} = e^{\varphi}\partial_0\partial_\alpha e^{-\varphi}\,, \quad a_{\alpha\beta} = e^{\varphi}\nabla_\alpha\nabla_\beta e^{-\varphi}\,, \\ b = (\partial_0\varphi)^2 - g^{\alpha\beta}(\partial_\alpha\varphi)(\partial_\beta\varphi)\,, \end{gathered}$$

where $\nabla_\alpha = \nabla_{\partial/\partial s^\alpha}$ is the covariant differentiation with respect to g.

Suppose now that $e^{2\varphi}h$ be flat on $U \times I$. Then, $\hat{R}_{jkrs} = 0$ for all j, k, r, s. Since $\bar{R}_{0\alpha\beta\gamma} = 0$, we have $a_{0\alpha} = 0$ for all α. So, $e^{-\varphi}$ is of the form

$$\psi = e^{-\varphi} = \psi_0(t) + \psi_1(s^1, \ldots, s^n)\,, \tag{28}$$

where ψ_0 is independent of $(s^1, \ldots, s^n)$ while ψ_1 is independent of t. Next, note that $\bar{R}_{0\alpha0\beta} = 0$ yields $(n^2 + n)/2$ equalities $a_{\alpha\beta} = (a_{00} - b)g_{\alpha\beta}$. From these and (28), we see that there exist two constants σ, τ such that

$$\begin{gathered} \psi_0'' + \sigma\psi_0 = \tau\,, \quad \nabla_\alpha\nabla_\beta\psi_1 + (\sigma\psi_1 + \tau)g_{\alpha\beta} = 0\,, \\ {\psi_0'}^2 + \sigma{\psi_0}^2 - 2\tau\psi_0 = g^{\alpha\beta}(\partial_\alpha\psi_1)(\partial_\beta\psi_1) + \sigma{\psi_1}^2 + 2\tau\psi_1\,. \end{gathered} \tag{29}$$

Furthermore, we have $\bar{R}_{\alpha\beta\gamma\delta} = -R_{\alpha\beta\gamma\delta}$. Then it follows that

$$R_{\alpha\beta\gamma\delta} = \sigma(g_{\alpha\delta}g_{\beta\gamma} - g_{\alpha\gamma}g_{\beta\delta})\,.$$

Therefore, the sectional curvature of g is constant and equal to σ.

Note that (29) yields the following equation for $\psi = e^{-\varphi}$.

$$(b_2)\qquad P_0[\psi^{(1-n)/2}] = -\sigma\left(\frac{n-1}{2}\right)^2\psi^{(1-n)/2}\,,$$

where $P_0 = \partial^2/\partial t^2 - \Delta_M$.

PROOF OF (ii). Suppose that there exists a coordinate system $(y^0, y^1, \ldots, y^n)$ and a smooth function a in $U \times I$ such that

$$P_c[fu] = e^{2\varphi}f\left\{\frac{\partial^2 u}{\partial(y^0)^2} - \frac{\partial^2 u}{\partial(y^1)^2} - \cdots - \frac{\partial^2 u}{\partial(y^n)^2}\right\}, \tag{30}$$

where $f = \psi^{(1-n)/2}a$. This implies first of all that $P_c f = 0$ and $P_c[fy^p] = 0$, that is,

$$P_c f = 0\,, \quad 2h^{lk}(\partial_l \log f)(\partial_k y^p) + P_0 y^p = 0\,, \quad 0 \le p \le n\,. \tag{31}$$

(30) also implies that $P_c[fy^p y^q] = 2\varepsilon^{pq}e^{2\varphi}f$, that is,

$$(\partial_j y^p)h^{jk}(\partial_k y^q) = e^{2\varphi}\varepsilon^{pq}\,, \quad e^{2\varphi}h_{jk} = (\partial_j y^p)\varepsilon_{pq}(\partial_k y^q)\,, \quad 0 \le j, k, p, q \le n\,, \tag{32}$$

where $\varepsilon^{00} = \varepsilon_{00} = 1$, $\varepsilon^{\alpha\alpha} = \varepsilon_{\alpha\alpha} = -1$ and $\varepsilon^{pq} = \varepsilon_{pq} = 0$ for $p \neq q$. (30) holds for any u if and only if (31) and (32) hold. We multiply $\varepsilon_{pq}\partial_j y^q$ to both sides of the p-th equation of (31) and then contract. Then, by a repeated use of (32), we have

$$\begin{aligned} -2e^{2\varphi}\partial_j \log f &= \varepsilon_{pq}(\partial_j y^p)\bar{\nabla}_k(h^{kl}\partial_l y^q) \\ &= \bar{\nabla}_k\{h^{kl}(\partial_j y^p)\varepsilon_{pq}(\partial_l y^q)\} - (1/2)\bar{\nabla}_j\{h^{kl}(\partial_k y^p)\varepsilon_{pq}(\partial_l y^q)\} \\ &= \bar{\nabla}_k(e^{2\varphi}h^{kl}h_{jl}) - (1/2)\bar{\nabla}_j(e^{2\varphi}h^{kl}h_{kl}) = (1-n)e^{2\varphi}\partial_j\varphi\,. \end{aligned}$$

So, $\partial_j a = 0$. Therefore, a is a (positive) constant. $P_c f = 0$ means now $P_c[\psi^{(1-n)/2}] = 0$. This together with (b_2) yields $c = \sigma(n-1)^2/4$.

Conversely, suppose that $c = \sigma(n-1)^2/4$. Let us confine ourselves to the case where $\sigma = 1, -1$ or 0. Then, if moreover U is small, (U, g) is isometric to an open subset of S^n, $\boldsymbol{H}^n$ or of $\boldsymbol{R}^n$, respectively (see Eisenhart [E, p. 85]).

If $\sigma = 0$, U is an open subset of $\boldsymbol{R}^n$ and if $(s^1, \dots, s^n)$ is a standard coordinate system of $\boldsymbol{R}^n$, then P_c $(c = 0)$ is naturally trivial. So, $\psi = 1$ $(\varphi = 0)$ is a solution to (29) with $\tau = 0$ and $y^j = s^j$, $0 \leq j \leq n$, satisfy (30).

For $\sigma = 0$, one of solutions to (29) with $\tau = 2$ is $\psi = \gamma(s)$, where $\gamma(s) = \sum_{p,q=0}^n \varepsilon_{pq}s^p s^q$. Changing letters, we define $y^j = x^j/\gamma(x)$, $0 \leq j \leq n$. Then we have the following equality for u with support in the domain $\{\gamma(x) > 0\}$ (note that $\gamma(x)\gamma(y) = 1$).

$$\begin{aligned} &\gamma(x)^{(n+3)/4} \sum_{p,q=0}^n \varepsilon^{pq} \frac{\partial^2}{\partial x^p \partial x^q}[\gamma(x)^{(1-n)/4}u] \\ &\qquad = \gamma(y)^{(n+3)/4} \sum_{p,q=0}^n \varepsilon^{pq} \frac{\partial^2}{\partial y^p \partial y^q}[\gamma(y)^{(1-n)/4}u]\,. \end{aligned} \tag{33}$$

This is nothing but the reflection principle in the Minkowski space-time $\boldsymbol{R}^n \times \boldsymbol{R}$.

For the proof of Lemma 2, it remains only to verify (18) and (22).

PROOF OF (18). $(\omega_1, \dots, \omega_n)$ in Section 3, (2°) is a coordinate system in the hemisphere $U = \{\omega_0 > 0\}$. $\psi = \omega_0 + \cos t$ is positive in $U \times (-\pi/2, \pi/2)$ and satisfies (29) $(\sigma = 1, \tau = 0)$. Let us define y^j's to be

$$y^j = -X^j \log\psi\,, \quad 0 \leq j \leq n\,,$$

where $X^0 = \partial/\partial t$, $X^\alpha = \omega_0\partial/\partial\omega_\alpha$ (see (17)). X^j's commute with P_c, and equations $P_c f = 0$, $X^j P_c f = 0$, $X^j X^k P_c f = 0$ $(f = \psi^{(1-n)/2})$ reduce to (31), (32). Hence we have (18).

PROOF OF (22). $(p_1, \dots, p_n)$ in Section 3, (3°) is a coordinate system valid everywhere in the hyperbolic space $\boldsymbol{H}^n$. $\psi = p_0 + \cosh t$ is positive and satisfies (29) $(\sigma = -1, \tau = 0)$. Let us define y^j's to be $y^0 = \partial(\log\psi)/\partial t$, $y^\alpha = p_0\partial(\log\psi)/\partial p_\alpha$ (see (21)). Then we have (31), (32) and (22). q.e.d.

6. Proof of (10), (11) and related remarks. Throughout this section, we denote simply $\alpha = \sqrt{a_1{}^2 + \cdots + a_k{}^2}$.

6.1. Proof of (10), (11). A power series expansion yields

$$\frac{\sin(\alpha t)}{\alpha} = \sum A_\lambda \prod_{j=1}^{k} \frac{(-a_j^2)^{\lambda_j}}{(2\lambda_j)!}, \quad \text{where} \quad A_\lambda = \frac{n!\, t^{2n+1}}{(2n+1)!} \prod_{j=1}^{k} \frac{(2\lambda_j)!}{\lambda_j!}.$$

Here, the summation on the right hand side is extended over all non-negative integers $\lambda_1, \ldots, \lambda_k$ and n stands for $\sum_{j=1}^{k} \lambda_j$. Then, A_λ splits into a product $A_\lambda = A_\lambda^{(1)} A_\lambda^{(2)}$, where

$$A_\lambda^{(1)} = \frac{\Gamma(n+(k/2))t^{2n+1}}{4\pi^{(k-1)/2}\Gamma(n+(3/2))} = \alpha_k \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^h t^{2n+k-2},$$
$$A_\lambda^{(2)} = \frac{2}{\Gamma(n+(k/2))} \prod_{j=1}^{k} \Gamma\left(\lambda_j + \frac{1}{2}\right).$$

$A_\lambda^{(2)}$ is equal to an integral over a simplex of dimension $k-1$:

$$A_\lambda^{(2)} = 2 \int \cdots \int_{s_j>0, s_1+\cdots+s_k=1} \prod_{j=1}^{k} s_j^{\lambda_j-(1/2)} ds_2 \cdots ds_k.$$

We rewrite it by setting $s_j = \omega_j^2$, $1 \le j \le k$. Since $2^{1-k} ds_2 \cdots ds_k / \sqrt{s_1 \cdots s_k}$ is the volume element of S^{k-1} and $\prod_{j=1}^{k} \omega^{2\lambda_j}$ is an even function on the sphere, we have finally

$$A_\lambda^{(2)} = \int \cdots \int_{S^{k-1}} \prod_{j=1}^{k} \omega_j^{2\lambda_j} dS_\omega.$$

Therefore, we have (10).

For the proof of (11), we proceed analogously to the above.

$$\frac{\sin(\alpha t)}{\alpha} = \sum C_\lambda \prod_{j=1}^{k} \frac{(-a_j^2)^{\lambda_j}}{(2\lambda_j)!}, \quad \text{where} \quad C_\lambda = \frac{n!\, t^{2n+1}}{(2n+1)!} \prod_{j=1}^{k} \frac{(2\lambda_j)!}{\lambda_j!},$$

and $n = \sum_{j=1}^{k} \lambda_j$ as above. C_λ splits as $C_\lambda = C_\lambda^{(1)} C_\lambda^{(2)}$:

$$C_\lambda^{(1)} = \frac{\Gamma(n+(k+1)/2)t^{2n+1}}{2\pi^{k/2}\Gamma(n+(3/2))} = \beta_k \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^h t^{2n+k-1},$$
$$C_\lambda^{(2)} = \frac{\sqrt{\pi}}{\Gamma(n+(k+1)/2)} \prod_{j=1}^{k} \Gamma\left(\lambda_j + \frac{1}{2}\right).$$

We represent $C_\lambda^{(2)}$ as an integral over a k-dimensional simplex and rewrite it as an integral over the ball B^k to have

$$C_\lambda^{(2)} = \int \cdots \int_{s_j>0, s_1+\cdots+s_k<1} \prod_{j=1}^{k} s_j^{\lambda_j-(1/2)} \frac{ds_1 \cdots ds_k}{\sqrt{1-s_1-\cdots-s_k}}$$
$$= \int \cdots \int_{B^k} \prod_{j=1}^{k} u_j^{2\lambda_j} \frac{du}{\sqrt{1-|u|^2}} .$$

Therefore, we have (11). q.e.d.

6.2. Derivation of (10), (11) from (15), (16). We set

$$u(p,t) = f(p)\frac{\sin(\alpha t)}{\alpha}, \quad f(p) = \cos(a_1 p_1)\cdots\cos(a_k p_k), \qquad p = (p_1, \dots, p_k) .$$

u solves the Cauchy problem (1)–(2) for the d'Alembertian of $k+1$ variables for this initial value f because $\sum_{j=1}^{k} \partial^2 f/\partial {p_j}^2 = -\alpha^2 f$. So, (15) or (16) holds according to the parity of k if $k \geq 2$. By setting $p = 0$, we have immediately (10) or (11), respectively.

We have shown that (10) implies (7) (see the proof of Lemma 1), (7) combined with (14) implies (15) (see Section 3, (1°)) and that (15) implies (10) as above. Therefore, (7), (10) and (15) are equivalent. Analogously, (8), (11) and (16) are equivalent.

6.3. Variants of (7), (8). Let us remark that

$$-\frac{1}{t}\frac{\partial}{\partial t}\left(\frac{1}{a_j}\frac{\partial}{\partial a_j}\frac{\sin(\alpha t)}{\alpha}\right) = \frac{\sin(\alpha t)}{\alpha}, \quad 1 \leq j \leq k . \tag{34}$$

In (10) or (11), we make use of this for $j = 1, \dots, p$, where $1 \leq p \leq k$.

If k is odd and $k \geq 3$, we have

$$\frac{\sin(\alpha t)}{\alpha} = \alpha_k \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{h+p} \left[t^{k-2+p} \int \cdots \int_{S^{k-1}} \left\{ \prod_{j=1}^{p} \frac{\omega_j \sin(a_j \omega_j t)}{a_j} \right\} \times \prod_{l=p+1}^{k} \cos(a_l \omega_l t) dS_\omega \right]. \tag{35}$$

Therefore, we have a variant of (7):

$$G\left(p, t, q; \sum_{j=1}^{k} c^{(j)}\right) = \alpha_k \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{h+p} \times \left[t^{k-2+p} \int \cdots \int_{S^{k-1}} \prod_{j=1}^{p} \{\omega_j G^{(j)}(p^{(j)}, \omega_j t, q^{(j)}; c^{(j)})\} \times \prod_{l=p+1}^{k} \frac{\partial G^{(l)}}{\partial t}(p^{(l)} \omega_l t, q^{(l)}; c^{(l)}) dS_\omega \right]. \tag{36}$$

Analogously, if k is even and $k \geq 2$, we have a variant of (8):

$$G\left(p, t, q; \sum_{j=1}^{k} c^{(j)}\right) = \beta_k \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{h+p}$$

$$\tag{37} \times \left[t^{k-1+p} \int \cdots \int_{B^k} \prod_{j=1}^{p} \{u_j G^{(j)}(p^{(j)}, u_j t, q^{(j)}; c^{(j)})\} \right.$$

$$\left. \times \prod_{l=p+1}^{k} \frac{\partial G^{(l)}}{\partial t}(p^{(l)}, u_l t, q^{(l)}; c^{(l)}) \frac{du}{\sqrt{1-|u|^2}} \right].$$

(36) or (37) may be simpler than (7) or (8), respectively, in some of applications.

6.4. On Remark 2 in Section 2. Set

$$P_1 = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial {p_1}^2}, \quad P_2 = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial {p_2}^2} - \frac{\partial^2}{\partial {p_3}^2}, \quad P_3 = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial {p_1}^2} - \frac{\partial^2}{\partial {p_2}^2} - \frac{\partial^2}{\partial {p_3}^2}.$$

P_3 is a Huygens operator although neither P_1 nor P_2 is. To verify this, we assume (14), (16) for $n = 2$ and prove (15) for $n = 3$ with the aid of (37) for $k = 2$, $p = 1$.

Let u be the solution to the Cauchy problem (1)–(2) for P_3 with initial value f. From (14), (16) and (37), it is evaluated at $p = o = (0, 0, 0)$ as follows.

$$u(o, t) = \frac{1}{4\pi t}\frac{\partial}{\partial t} L(t),$$

where

$$L(t) = \frac{t^3}{\pi} \iint_{B^2} \frac{\mu^2 d\lambda d\mu}{\sqrt{1-\lambda^2-\mu^2}} \iint_{B^2} \frac{f(\lambda t, \mu q_2 t, \mu q_3 t)}{\sqrt{1-{q_2}^2-{q_3}^2}}\, dq_2 dq_3.$$

We restrict the domain of integration with respect to μ to the part $\mu > 0$. For fixed μ, we change the variables from (λ, q_2, q_3) to $(x_1, x_2, x_3) = (\lambda, \mu q_2, \mu q_3)$. Then we have

$$L(t) = \frac{t^3}{\pi} \iiint_{B^3} f(tx)dx \int_{\sqrt{{x_2}^2+{x_3}^2}}^{\sqrt{1-{x_1}^2}} \frac{2\mu d\mu}{\sqrt{1-{x_1}^2-\mu^2}\sqrt{\mu^2-{x_2}^2-{x_3}^2}}.$$

The integral with respect to μ is equal to π for every x, so

$$L(t) = t^3 \iiint_{B^3} f(tx)dx \quad \text{and} \quad u(o, t) = \frac{t}{4\pi} \iint_{S^2} f(t\omega) dS_\omega.$$

The last equality is precisely (15) at $p = o$ for $n = 3$.

6.5. On Remark 1 in Section 2. Suppose that (13) holds for a Φ with support in S^{k-1}. Since $\sin\alpha/\alpha$ is an even function of a_j for every j, we can replace Φ by its even part Φ_e defined to be

$$\langle \Phi_e, f \rangle = \left\langle \Phi, 2^{-k} \sum f(\varepsilon_1 p_1, \varepsilon_2 p_2, \ldots, \varepsilon_k p_k) \right\rangle$$

for every test function $f \in C^\infty(\boldsymbol{R}^k)$, where the summation is extended over 2^k terms with $\varepsilon_j = +1$ or -1. Φ_e has also support in S^{k-1}. Next, we can replace $\cos(a_j p_j)$ by $e^{ia_j p_j}$ because $\sin(a_j p_j)$ is an odd function. So, (13) is rewritten as

$$(c_1) \qquad \frac{\sin\alpha}{\alpha} = \langle \Phi_e, e^{i\langle a,p\rangle}\rangle, \quad \text{where} \quad \langle a,p\rangle = \sum_{j=1}^{k} a_j p_j .$$

Since $\sin\alpha/\alpha$ is invariant by rotations $a \mapsto Ua$ ($U \in SO(k)$) and $\langle Ua, p\rangle = \langle a, {}^t Up\rangle$, we can replace Φ_e by its average Ψ over $SO(k)$ defined to be

$$\langle \Psi, f\rangle = \left\langle \Phi_e, \int_{SO(k)} f({}^t Up)dU \right\rangle$$

for every f, where dU is the Haar measure with total mass 1 on $SO(k)$. So, (c_1) is rewritten as

$$(c_2) \qquad \frac{\sin\alpha}{\alpha} = \langle \Psi, e^{i\langle a,p\rangle}\rangle .$$

Now, Ψ has support in S^{k-1} and is a rotation invariant distribution on $\boldsymbol{R}^k$, that is, $\langle \Psi, (d/ds)\{f(e^{-sX}p)\}\rangle|_{s=0} = 0$ for every f and every anti-symmetric real matrix X of order k. Hence there exists a polynomial q of single variable such that

$$\langle \Psi, f\rangle = \int_{S^{k-1}} \left\{ q\left(r\frac{\partial}{\partial r}\right) f(r\omega)\right\}\Big|_{r=1} dS_\omega = \left\{ q\left(r\frac{\partial}{\partial r}\right)\int_{S^{k-1}} f(r\,\omega)dS_\omega\right\}\Big|_{r=1}$$

for every f. This and (c_2) imply

$$(c_3) \qquad \frac{\sin\alpha}{\alpha} = \left\{ q\left(r\frac{\partial}{\partial r}\right)\int_{S^{k-1}} e^{ir\langle a,\omega\rangle} dS_\omega\right\}\Big|_{r=1} .$$

By a computation analogous to that in §6.1, we have

$$\int_{S^{k-1}} e^{i\langle a,\omega\rangle} dS_\omega = (2\pi)^{\lambda+1}\alpha^{-\lambda}J_\lambda(\alpha) = 2\pi^{\lambda+1}\sum_{m=0}^{\infty}\frac{(-\alpha^2/4)^m}{m!\Gamma(m+\lambda+1)},$$

where $\lambda = (k-2)/2$ and J_λ is the Bessel function of order λ. This and (c_3) yield

$$(c_4) \qquad \frac{\sin\alpha}{\alpha} = (2\pi)^{\lambda+1}\left[q\left(r\frac{\partial}{\partial r}\right)\{(r\alpha)^{-\lambda}J_\lambda(r\alpha)\}\right]_{r=1} .$$

Comparing the coefficients of α^{2m} in the Taylor expansions of both sides, we obtain a system of an infinite number of equations

$$(c_5) \qquad q(2m) = \frac{\pi^{(1-k)/2}}{4}\frac{\Gamma(m+(k/2))}{\Gamma(m+(3/2))} \quad \text{for } m = 0, 1, 2, \ldots .$$

(13) holds for a Φ with support in S^{k-1} if and only if (c_5) holds with a polynomial q.

If k is odd, (c_5) defines a polynomial q of degree $(k-3)/2$. If on the contrary k is even, there does not exist such a polynomial q because the right hand side of (c_5) behaves like a positive number multiple of $m^{(k-3)/2}$ as $m \to \infty$ due to the Stirling formula. q.e.d.

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