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FUNCTION SPACES AND CLASSES OF PSEUDODIFFERENTIAL OPERATORS

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Abstract. We introduce a new class of selfadjoint compact pseudodifferential operators, which is analogous to a class of elliptic unbounded pseudodifferential operators and is, therefore, suitable for obtaining upper and lower estimates on the eigenvalues of operators in this class. We prove such estimates and, as an application, we show that any operator from this class belongs to the Schatten-von Neuman class if and only if its symbol belongs to the Lorentz space.

1. Introduction. For a symbol $\sigma \in \mathcal{C}^\infty(\boldsymbol{R}^{2d})$ having derivatives with a common polynomial growth, we define a *pseudodifferential operator* $\sigma(D, x)$, according to the Weyl calculus, to be

$$\sigma(D,x)f(x) = \iint e^{2\pi i(x-y)\xi}\sigma\left(\xi, \frac{x+y}{2}\right)f(y)dyd\xi$$

for f belonging to the Schwartz class $S(\boldsymbol{R}^d)$. Thus $\sigma(D, x)$ is a continuous operator from $S(\boldsymbol{R}^d)$ to $S(\boldsymbol{R}^d)$ and extends to a continuous operator from $S'(\boldsymbol{R}^d)$ to $S'(\boldsymbol{R}^d)$ (see [F]), where $S'(\boldsymbol{R}^d)$ denotes the space of tempered distributions. If σ is real, then $\sigma(D, x)$ is a symmetric operator. Moreover, certain growth restrictions on the symbol σ ensure that the closure of $\sigma(D, x)$ is a selfadjoint operator (see [G]).

There are two ways to guarantee that a selfadjoint operator $\sigma(D, x)$ has a discrete spectrum. If σ has bounded derivatives and $\lim_{|z|\to\infty}\sigma(z) = 0$, then $\sigma(D, x)$ is compact, therefore diagonalizable (see [H]). The other way is to assume that $\lim_{|z|\to\infty}\sigma(z) = \infty$, and that σ is elliptic, that is, it satisfies certain conditions on the growth of its derivatives (see the next section for details). In this case we obtain an unbounded diagonalizable operator, which can be thought of as an inverse of a compact operator. The theory of such unbounded pseudodifferential operators is well developed (see for example [F], [H2]). In particular, many researchers obtained estimates on eigenvalues of such operators, which permitted studies of the asymptotic behavior of their spectra. A standard example is the Weyl asymptotic formula, which approximates the number of eigenvalues of $\sigma(D, x)$ smaller than λ by the area of a set $\{z \in \boldsymbol{R}^{2d}; \sigma(z) \le \lambda\}$:

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$$N(\lambda) = \int_{\sigma \le \lambda} dz + O(\lambda^{-\rho}),$$

where $\rho > 0$ depends on the symbol σ (see [H1], [G], [CR1]).

Another approximation of eigenvalues of $\sigma(D, x)$ can be achieved with the use of a non-decreasing rearrangement of $\{\sigma(am, bn)_{n,m\in \mathbf{Z}^d}\}$, where $a, b > 0$ (see [T], [RT], [CR2]).

The case of compact pseudodifferential operators was approached in a slightly different manner. Mostly, singular values instead of eigenvalues were studied. Moreover, researchers were looking for minimal conditions on the symbol σ, which would allow estimates from above of the singular values of $\sigma(D, x)$ (see [R], [HRT], [RT]). This theme is very strong and goes back to the Calderón-Vaillancourt theorem and the problem of finding the weakest possible conditions on σ that ensure that $\sigma(D, x)$ is bounded on $L^2(\mathbf{R}^d)$ (see [HRT]).

In this note we introduce a class of elliptic compact selfadjoint pseudodifferential operators which originates from a class of elliptic unbounded selfadjoint pseudodifferential operators. Then we use Beals's theory on powers of pseudodifferential operators (see [B], [CR1]) to translate already known results about spectral asymptotics for the latter class to the new setting. As an application of such estimates on eigenvalues of $\sigma(D, x)$ we show that, within the introduced class, $\sigma(D, x)$ belongs to the Schatten-von Neumann class $S_{p,q}$ if and only if σ belongs to the Lorentz space $L^{p,q}(\mathbf{R}^{2d})$.

We would like to mention that the result on spectral estimates of pseudodifferential operators (Theorem 3.1) depends on the Gabor expansions of their symbols. That is, we use collections of the form $\{g_{m,n}(x) = e^{-2\pi i am\cdot x} g(x - bn); m, n \in \mathbf{Z}^d\}$, where $g \in L^2(\mathbf{R}^d)$, and $a, b > 0$. For more details on this subject we refer the reader to [FS] and [G1].

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2. Preliminaries. A positive continuous function w on $\mathbf{R}^{2d}$ is called a *weight* if

$$w(z + z') \le C(1 + |z'|)^{\gamma} w(z)$$

for every $z, z' \in \mathbf{R}^{2d}$ and some positive constants C, γ. Furthermore, we say that a weight w is *smooth* if $w \in \mathcal{C}^\infty(\mathbf{R}^{2d})$. The set of weights is a group under multiplication. Moreover, if $\beta \in \mathbf{R}$ and w is a weight, then w^β is also a weight. For a fixed weight w, we define $S(w)$ to be the set of all functions $f \in \mathcal{C}^\infty(\mathbf{R}^{2d})$ such that for every $\alpha \in \mathbf{N}_0^{2d}$, there exists $C_\alpha > 0$ such that

$$|\partial^\alpha f(z)| \le C_\alpha w(z)$$

for all $z \in \mathbf{R}^{2d}$. (In our convention, the set of natural numbers $\mathbf{N}$ does not contain zero, and $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$.) For multiindices $\alpha, \beta \in \mathbf{N}_0^{2d}$, we say that $\beta \le \alpha$ if $\beta_i \le \alpha_i$ for all i. We also define the length of α to be $|\alpha| = \sum_i \alpha_i$.

DEFINITION 2.1. Let $0 < \tau \leq 1$. We say that a smooth weight w is τ^--*elliptic* if $\lim_{|z|\to\infty} w(z) = \infty$ and for every $\alpha \in \boldsymbol{N}_0^{2d} \setminus \{0\}$, there is a $C_\alpha > 0$ such that

$$|\partial^\alpha w(z)| \leq C_\alpha w(z)^{1-\tau}$$

for all $z \in \boldsymbol{R}^{2d}$. Similarly, for $0 < \tau \leq 1$ we say that a smooth weight w is τ^+-*elliptic* if $\lim_{|z|\to\infty} w(z) = 0$ and for every $\alpha \in \boldsymbol{N}_0^{2d} \setminus \{0\}$, there is a $C_\alpha > 0$ such that

$$|\partial^\alpha w(z)| \leq C_\alpha w(z)^{1+\tau}$$

for all $z \in \boldsymbol{R}^{2d}$.

The following proposition explains the origin of the class of τ^+-elliptic weights.

PROPOSITION 2.1 *A smooth weight w is τ^--elliptic if and only if $1/w$ is τ^+-elliptic.*

PROOF. Let w be a τ^--elliptic weight. We will use induction over $n = |\alpha|$ to prove that $1/w$ is τ^+-elliptic. Let $\gamma \in \boldsymbol{N}_0^{2d}$ such that $|\gamma| = 1$. We have

$$\left|\partial^\gamma\left(\frac{1}{w}\right)\right| = \left|\frac{\partial^\gamma w}{w^2}\right| \leq C_\gamma \frac{w^{1-\tau}}{w^2} = C\gamma\left(\frac{1}{w}\right)^{1+\tau}$$

for some $C_\gamma > 0$. Let us assume that $|\partial^\alpha(1/w)| \leq C_\alpha(1/w)^{1+\tau}$ for all $0 < |\alpha| \leq n$. We claim that for such α,

$$\left|\partial^\alpha\left(\frac{1}{w^2}\right)\right| \leq C_\alpha \frac{1}{w^2}\,. \tag{2.1}$$

In fact, to validate (2.1), we use the Leibniz rule:

$$\left|\partial^\alpha\left(\frac{1}{w^2}\right)\right| \leq \sum_{0\leq\beta\leq\alpha} C_\beta\left|\partial^\beta\left(\frac{1}{w}\right)\partial^{\alpha-\beta}\left(\frac{1}{w}\right)\right| \leq C_\alpha\left(\left(\frac{1}{w}\right)^{2+2\tau} + \left(\frac{1}{w}\right)^{2+\tau}\right) \leq C_\alpha\frac{1}{w^2}\,,$$

where the last inequality follows from the fact that $\lim_{|z|\to\infty} w(z) = \infty$. Therefore, by (2.1), we obtain for $|\alpha| = n$ and $|\gamma| = 1$:

$$\left|\partial^\alpha\partial^\gamma\left(\frac{1}{w}\right)\right| = \left|\partial^\alpha\left(\frac{\partial^\gamma w}{w^2}\right)\right| \leq \sum_{0\leq\beta\leq\alpha} C_\beta\left|\partial^\beta(\partial^\gamma w)\partial^{\alpha-\beta}\left(\frac{1}{w^2}\right)\right| \leq C_\alpha\frac{w^{1-\tau}}{w^2} = C_\alpha\left(\frac{1}{w}\right)^{1+\tau}.$$

Thus, by induction, the weight $1/w$ is τ^+-elliptic.

To prove the other implication we denote $1/w$ by u and repeat the above reasoning to show that $1/u$ is τ^--elliptic. For $|\gamma| = 1$ we obtain $|\partial^\gamma(1/u)| \leq C_\gamma(1/u)^{1-\tau}$. If we assume that $|\partial^\alpha(1/u)| \leq C_\alpha(1/u)^{1-\tau}$ for $0 < |\alpha| \leq n$, then from the Leibniz rule it follows that, for such α, we have $|\partial^\alpha(1/u^2)| \leq C_\alpha(1/u^2)$, since $\lim_{|z|\to\infty} u(z) = 0$. Thus, for $|\alpha| = n$ and $|\gamma| = 1$, we obtain $|\partial^\alpha\partial^\gamma(1/u)| \leq C_\alpha(1/u)^{1-\tau}$, and, by induction, the weight $1/u$ is τ^--elliptic. □

For the purpose of this note, we will arrange the eigenvalues $\{\lambda_k\}_{k\in\boldsymbol{N}}$ of a compact selfadjoint operator in a decreasing order of their absolute values, i.e., $|\lambda_1| \geq |\lambda_2| \geq \cdots$. However, the eigenvalues $\{\lambda_k\}_{k\in\boldsymbol{N}}$ of an unbounded selfadjoint operator with discrete spectrum, which is bounded below, will be arranged in a non-decreasing order, that is, $\lambda_1 \leq \lambda_2 \leq \cdots$.

For a measurable real-valued function f on $\boldsymbol{R}^{2d}$, we define its *non-increasing rearrangement* f^* by

$$f^*(t) = \inf\{\lambda \in \boldsymbol{R};\ |\{x \in \boldsymbol{R}^{2d};\ |f(x)| > \lambda\}| \leq t\}$$

for $0 < t < \infty$.

The *Lorentz space* $L^{p,q}(\boldsymbol{R}^{2d})$, where $0 < p < \infty$, $0 < q \leq \infty$, is the set of all measurable functions f on $\boldsymbol{R}^{2d}$ that satisfy

$$\int_0^\infty t^{q/p-1} f^*(t)^q dt < \infty\,, \quad q < \infty\,,$$

and

$$\sup_{t>0} t^{1/p} f^*(t) < \infty\,, \quad q = \infty\,.$$

If $\{c_{m,l}\}_{m,l\in\boldsymbol{Z}^d}$ is a sequence of real numbers, then we denote the non-increasing rearrangement of $\{|c_{m,l}|\}_{m,l\in\boldsymbol{Z}^d}$, if it exists, by $\{c_k^*\}_{k\in\boldsymbol{N}}$. For $0 < p < \infty$, $0 < q \leq \infty$, we define a discrete version of the Lorentz space, $L_d^{p,q}(\boldsymbol{Z}^{2d})$, to be the set of all sequences $\{c_{m,l}\}_{m,l\in\boldsymbol{Z}^d}$ that satisfy

$$\sum_{k=1}^\infty k^{q/p-1}(c_k^*)^q < \infty\,, \quad q < \infty\,,$$

and

$$\sup_{k\in\boldsymbol{N}} k^{1/p} c_k^* < \infty\,, \quad q = \infty\,.$$

We say that a compact selfadjoint operator belongs to the *Schatten-von Neumann class* $S_{p,q}$, where $0 < p < \infty$, $0 < q \leq \infty$, if its eigenvalues $\{\lambda_k\}_{k\in\boldsymbol{N}}$ satisfy

$$\sum_{k=1}^\infty k^{q/p-1}|\lambda_k|^q < \infty\,, \quad q < \infty\,,$$

and

$$\sup_{k\in\boldsymbol{N}} k^{1/p}\lambda_k < \infty\,, \quad q = \infty\,.$$

3. Estimates on eigenvalues of compact pseudodifferential operators. In his paper [T], K. Tachizawa showed that a class of elliptic unbounded pseudodifferential operators can be approximately diagonalized in the Wilson basis. This result was repeated in [RT] for the same class of operators and local trigonomertic bases. A similar diagonalization was proven in [CR2] for a more general class of operators (whose symbols are τ^--elliptic weights) where Gabor frames, instead of orthonormal bases, were used. In particular, the approximate diagonalization result allows us to estimate the eigenvalues of such operators by a non-decreasing rearrangement of $\{\sigma(am, bn)\}_{m,n\in\boldsymbol{Z}^d}$, $a, b > 0$. To be more precise, let us cite the following result of [CR2]:

THEOREM 3.1. *Suppose that the symbol σ is a τ^--elliptic weight. Then $\sigma(D, x)$ is an unbounded selfadjoint operator with discrete spectrum $\lambda_1 \leq \lambda_2 \leq \cdots$, and there exist* *positive constants* $C(a,b)$ *and* $C(c,d)$ *such that for large enough* $k \in \boldsymbol{N}$

$$\mu_k^{a,b} - C(a,b)\left(\mu_k^{a,b}\right)^{1-\tau} \le \lambda_k \le \mu_k^{c,d} + C(c,d)\left(\mu_k^{c,d}\right)^{1-\tau},$$

where a, b, c, d *are positive constants, satisfying* $ab < 1$, $cd > 1$, *and the sequence* $\{\mu_k^{a,b}\}_{k\in\boldsymbol{N}}$ *is the non-decreasing rearrangement of* $\{\sigma(am, bn)\}_{m,n\in\boldsymbol{Z}^d}$, *and, similarly,* $\{\mu_k^{c,d}\}_{k\in\boldsymbol{N}}$ *is the non-decreasing rearrangement of* $\{\sigma(cm, dn)\}_{m,n\in\boldsymbol{Z}^d}$.

The above theorem can be used to prove similar estimates on eigenvalues of operators whose symbols are τ^+-elliptic weights. In fact, Proposition 2.1 indicates that such operators can be viewed (at least intuitively) as inverses of elliptic unbounded operators coming from τ^--elliptic weights. Therefore, all estimates on eigenvalues of the latter can be transformed into analogous estimates on the eigenvalues of operators whose symbols are τ^+-elliptic weights. However, the formal arguments, which we shall present below, require Beals's theory on powers of pseudodifferential operators (see [B] for the Kohn-Nirenberg calculus, or [CR1] for the Weyl calculus).

THEOREM 3.2. *Suppose that the symbol* σ *is a* τ^+*-elliptic weight. Then* $\sigma(D,x)$ *is a compact selfadjoint operator with eigenvalues* $\{\lambda_k\}_{k\in\boldsymbol{N}}$, *and there exist positive constants* $C(a,b)$ *and* $C(c,d)$ *such that for* $k \in \boldsymbol{N}$ *large enough*

$$C(c,d)\mu_k^{c,d} \le \lambda_k \le \mu_k^{a,b} + C(a,b)(\mu_k^{a,b})^{1+\tau},$$

where a, b, c, d *are positive constants, satisfying* $ab < 1$, $cd > 1$, *and the sequence* $\{\mu_k^{a,b}\}_{k\in\boldsymbol{N}}$ *is the non-increasing rearrangement of* $\{\sigma(am, bn)\}_{m,n\in\boldsymbol{Z}^d}$, *and, similarly,* $\{\mu_k^{c,d}\}_{k\in\boldsymbol{N}}$ *is the non-increasing rearrangement of* $\{\sigma(cm, dn)\}_{m,n\in\boldsymbol{Z}^d}$.

PROOF. For the purpose of this proof, given a smooth weight w, we will introduce a new class of symbols, denoted by $S(w)$, to be the set of all functions $a \in \mathcal{C}^\infty(\boldsymbol{R}^{2d})$ such that for all $k \in \boldsymbol{N}_0$:

$$\max_{|\alpha|\le k} \|w^{-1}\partial^\alpha a\|_\infty < \infty .$$

As mentioned before, the fact that σ is real and decays to zero at infinity implies that $\sigma(D,x)$ is a compact selfadjoint operator. Therefore, we will concentrate only on proving the estimates on its eigenvalues.

By Proposition 2.1 it is immediate that the symbol $1/\sigma$ satisfies assumptions of Theorem 3.1, thus, there exists $K > 0$ such that the operator $(1/\sigma)(D,x) + K$ is positive. Let $\delta = (1/\sigma) + K$. By Corollary 2.4 of [CR1] we obtain $\delta^{\circ-1} = 1/\delta + r_{-1}$, where $r_{-1} \in S(\delta^{-1-\tau})$ and $\delta^{\circ-1}$ denotes the symbol of $\delta(D,x)^{-1}$.

We claim that

$$\delta^{\circ-1} = \sigma + r, \tag{3.1}$$

where $r \in S(\delta^{-1-\tau})$. In fact, we have $1/\delta = \sigma - K\sigma^2/(1+K\sigma)$. Therefore, it is enough to prove that $\sigma^2/(1+K\sigma) = \sigma/\delta \in S(\delta^{-2}) \subset S(\delta^{-1-\tau})$. Since σ is a τ^+-elliptic weight, from Proposition 2.1 it follows that δ is a τ^--elliptic weight. Thus, using Proposition 2.1 again, we obtain that $1/\delta$ is a τ^+-elliptic weight, which implies that $1/\delta \in S(\delta^{-1})$. Moreover, since σ

is a τ^+-elliptic weight, we have $\sigma \in S(\sigma) \subset S(\delta^{-1})$, where the last inclusion follows from $\sigma \leq M/\delta$ for some $M > 0$. To finish the proof of our claim, we observe that $\sigma, 1/\delta \in S(\delta^{-1})$ implies $\sigma/\delta \in S(\delta^{-2})$.

By (3.1) we have $\sigma(D,x) = \delta(D,x)^{-1} - r(D,x)$. From Corollary 2.6 in [CR1], it follows that there exists $L > 0$ such that

$$-L\delta(D,x)^{-1-\tau} \leq r(D,x) \leq L\delta(D,x)^{-1-\tau} .$$

Therefore

$$\delta(D,x)^{-1} - L\delta(D,x)^{-1-\tau} \leq \sigma(D,x) \leq \delta(D,x)^{-1} + L\delta(D,x)^{-1-\tau} . \tag{3.2}$$

From Proposition 2.1 it follows that δ satisfies the assumptions of Theorem 3.1, and hence we obtain for $k \in \boldsymbol{N}$ large enough

$$(\mu_k^{a,b})^{-1}+K-C(a,b)((\mu_k^{a,b})^{-1}+K)^{1-\tau} \leq \gamma_k \leq (\mu_k^{c,d})^{-1}+K+C(c,d)((\mu_k^{c,d})^{-1}+K)^{1-\tau} ,$$

where $\gamma_1 \leq \gamma_2 \leq \cdots$ are the eigenvalues of $\delta(D,x)$. From the above inequality it follows that we can enlarge the constants $C(a,b), C(c,d)$ to get

$$(\mu_k^{a,b})^{-1} - C(a,b)((\mu_k^{a,b})^{-1})^{1-\tau} \leq \gamma_k \leq (\mu_k^{c,d})^{-1} + C(c,d)((\mu_k^{c,d})^{-1})^{1-\tau}$$

for large enough $k \in \boldsymbol{N}$. Therefore, for such k's, we obtain

$$\mu_k^{c,d} - C(c,d)(\mu_k^{c,d})^{1+\tau} \leq \gamma_k^{-1} \leq \mu_k^{a,b} + C(a,b)(\mu_k^{a,b})^{1+\tau} . \tag{3.3}$$

Applying the "min-max" principle (see [SW]) to (3.2) yields for large enough $k \in \boldsymbol{N}$:

$$\frac{1}{C}\gamma_k^{-1} \leq \gamma_k^{-1} - L\gamma_k^{-1-\tau} \leq \lambda_k \leq \gamma_k^{-1} + L\gamma_k^{-1-\tau} ,$$

which, together with (3.3), gives us the desired estimates:

$$C(c,d)\mu_k^{c,d} \leq \lambda_k \leq \mu_k^{a,b} + C(a,b)(\mu_k^{a,b})^{1+\tau} . \qquad \square$$

4. Schatten classes. One of the first results concerning the Weyl calculus and the Hilbert-Schmidt class of operators was proven by J. Pool, [P]. There it is shown that the operator $\sigma(D,x)$ belongs to the Hilbert-Schmidt class if and only if its symbol σ belongs to $L^2(\boldsymbol{R}^2)$. Moreover, $\|\sigma(D,x)\|_{HS} = \|\sigma\|_2$. Other results related to pseudodifferential operators and Schatten classes S_p or more general Schatten-von Neumann classes $S_{p,q}$ can be found in [H], [HRT] and [RT]. In particular, from Theorem 5 in [RT] it follows that if the symbol σ is a τ^+-elliptic weight and $\sigma \in L^{p,q}(\boldsymbol{R}^{2d})$, then $\sigma(D,x) \in S_{p,q}$. We will use methods similar to those of [RT] to show that, in fact, one has equivalence of these conditions.

PROPOSITION 4.1. *Let σ be a weight and $a,b > 0$. If there exists a non-increasing rearrangement of $\{\sigma(am,bn)\}_{m,n\in\boldsymbol{Z}^d}$, then $\{\sigma(am,bn)\} \in L_d^{p,q}(\boldsymbol{Z}^{2d})$ if and only if $\sigma \in L^{p,q}(\boldsymbol{R}^{2d})$.*

PROOF. Let $\{\mu_k\}_{k\in\boldsymbol{N}}$ denote the non-increasing rearrangement of $\{\sigma(am,bn)\}_{m,n\in\boldsymbol{Z}^d}$, and let m_k, n_k be defined by $\mu_k = \sigma(am_k, bn_k)$ for $k \in \boldsymbol{N}^d$. Since σ is a weight, we have, for

(ξ, x) satisfying $|\xi - am_k| \le a/2$, $|x - bn_k| \le b/2$, that

$$\frac{1}{C}\mu_k \le \sigma(\xi, x) \le C\mu_k$$

for some positive C. Therefore, we obtain

$$\left|\left\{(\xi, x); \sigma(\xi, x) > \frac{1}{C}\mu_k\right\}\right| \ge ck$$

and

$$|\{(\xi, x); \sigma(\xi, x) > C\mu_k\}| \le ck\,,$$

where c is a positive constant depending only on a and b. This implies that there exists a constant $C > 0$ such that for every $k \in \boldsymbol{N}$,

$$\frac{1}{C}\mu_k \le \sigma^*(ck) \le C\mu_k\,.$$

Now, if $0 < p \le q < \infty$, we have

$$\begin{aligned}\int_0^\infty t^{q/p-1}\sigma^*(t)^q\,dt &= A\sum_{k=0}^\infty \int_k^{k+1} t^{q/p-1}\sigma^*(ct)^q\,dt \ge A\sum_{k=0}^\infty k^{q/p-1}\sigma^*(c(k+1))^q \\ &\ge A\sum_{k=0}^\infty k^{q/p-1}\mu_{k+1}^q \ge A\sum_{k=1}^\infty k^{q/p-1}\mu_k^q\,,\end{aligned}$$

and

$$\begin{aligned}\int_0^\infty t^{q/p-1}\sigma^*(t)^q\,dt &\le A\sigma^*(0)^q + A\sum_{k=1}^\infty (k+1)^{q/p-1}\sigma^*(ck)^q \\ &\le A\sigma^*(0)^q + A\sum_{k=1}^\infty k^{q/p-1}\mu_k^q\,,\end{aligned}$$

where A denotes a constant (that may possibly differ from line to line), which depends on the function σ. In the second sequence of inequalities, we have used the fact that $\sigma > 0$. Here, $\sigma^*(0) = \lim_{t\to 0^+}\sigma^*(t)$. The assumptions that σ is a weight, and that there exists a non-increasing rearrangement of $\{\sigma(am, bn)\}$, imply that $\sigma^*(0) < \infty$. If $0 < q < p < \infty$, then

$$\begin{aligned}\int_0^\infty t^{q/p-1}\sigma^*(t)^q\,dt &\ge A\sum_{k=0}^\infty (k+1)^{q/p-1}\sigma^*(c(k+1))^q \\ &\ge A\sum_{k=0}^\infty (k+1)^{q/p-1}\mu_{k+1}^q \ge A\sum_{k=1}^\infty k^{q/p-1}\mu_k^q\,,\end{aligned}$$

and

$$\int_0^\infty t^{q/p-1}\sigma^*(t)^q dt \le A\int_0^1 t^{q/p-1}\sigma^*(t)^q dt + A\sum_{k=1}^\infty k^{q/p-1}\sigma^*(ck)^q$$
$$\le A\sigma^*(0)^q + A\sum_{k=1}^\infty k^{q/p-1}\mu_k^q\,,$$

with the same remarks as above. If $q = \infty$, then

$$\sup_{k\in\boldsymbol{N}}\{k^{1/p}\mu_k\} \le A\sup_{k\in\boldsymbol{N}}\{k^{1/p}\sigma^*(ck)\} \le A\sup_{t>0}\{t^{1/p}\sigma^*(ct)\} \le A\sup_{t>0}\{t^{1/p}\sigma^*(t)\}\,,$$

and

$$\sup_{t>0}\{t^{1/p}\sigma^*(t)\} \le A\sup_{t>0}\{t^{1/p}\sigma^*(ct)\} \le A\sigma^*(0) + A\sup_{k\in\boldsymbol{N}}\{k^{1/p}\sigma^*(ck)\}$$
$$\le A\sigma^*(0) + A\sup_{k\in\boldsymbol{N}}\{k^{1/p}\mu_k\}\,.$$

□

THEOREM 4.2. *Suppose that the symbol σ is a τ^+-elliptic weight. Then the compact operator $\sigma(D,x) \in S_{p,q}$ if and only if $\sigma \in L^{p,q}(\boldsymbol{R}^{2d})$.*

PROOF. Let $\{\lambda_k\}_{k\in\boldsymbol{N}}$ be the set of eigenvalues of the operator $\sigma(D,x)$. For positive constants a, b, c, d, satisfying $ab < 1$, $cd > 1$, let the sequence $\{\mu_k^{a,b}\}_{k\in\boldsymbol{N}}$ denote the non-increasing rearrangement of $\{\sigma(am, bn)\}_{m,n\in\boldsymbol{Z}^d}$, and, similarly, let $\{\mu_k^{c,d}\}_{k\in\boldsymbol{N}}$ denote the non-increasing rearrangement of $\{\sigma(cm, dn)\}_{m,n\in\boldsymbol{Z}^d}$ (note that these rearrangements exist, because $\lim_{|z|\to\infty}\sigma(z) = 0$). Since σ satisfies the assumptions of Theorem 3.2, we have

$$0 < \frac{1}{C}\mu_k^{c,d} \le \lambda_k \le C\mu_k^{a,b} \tag{4.1}$$

for large enough $k \in \boldsymbol{N}$ and some $C > 0$. Thus, if $\sigma(D,x) \in S_{p,q}$, we obtain $\{\sigma(cm, dn)\} \in L_d^{p,q}(\boldsymbol{Z}^{2d})$, which, by Proposition 4.1, implies that $\sigma \in L^{p,q}(\boldsymbol{R}^{2d})$. On the other hand, if $\sigma \in L^{p,q}(\boldsymbol{R}^{2d})$, then from Proposition 4.1 it follows that $\{\sigma(am, bn)\} \in L_d^{p,q}(\boldsymbol{Z}^{2d})$, and hence (4.1) implies that $\sigma(D,x) \in S_{p,q}$. We would like to remind the reader once again that this implication has been proved in [RT]. □

Since, for $0 < p < \infty$, we have $L^{p,p}(\boldsymbol{R}^{2d}) = L^p(\boldsymbol{R}^{2d})$ and $S_{p,p} = S_p$, the usual Schatten class, we obtain the following

COROLLARY 4.3. *If the symbol σ is a τ^+-elliptic weight, and $0 < p < \infty$, then $\sigma(D,x) \in S_p$ if and only if $\sigma(\xi, x) \in L^p(\boldsymbol{R}^{2d})$.*

Theorem 3.1 and Proposition 4.1 allow us to obtain the following result related to Theorem 4.2, but concerning operators whose symbols are τ^--elliptic weights.

THEOREM 4.4. *Suppose that symbol σ is a τ^--elliptic weight. If the unbounded operator $\sigma(D,x)$ is invertible, then $\sigma(D,x)^{-1} \in S_{p,q}$ if and only if $1/\sigma \in L^{p,q}(\boldsymbol{R}^{2d})$.*

PROOF. We will use the same notation as in the proof of Theorem 4.2, with the only difference that $\{\mu_k^{a,b}\}_{k\in\boldsymbol{N}}$ and $\{\mu_k^{c,d}\}_{k\in\boldsymbol{N}}$ shall now denote the non-decreasing rearrangements (which exist, since $\lim_{|z|\to\infty}\sigma(z)=\infty$). From Theorem 3.1 it follows that

$$0 < \frac{1}{C}\mu_k^{a,b} \le \lambda_k \le C\mu_k^{c,d} \tag{4.2}$$

for $k \in \boldsymbol{N}$ large enough and some $C > 0$. Assume that $\sigma(D,x)^{-1} \in S_{p,q}$. Since $\{(\mu_k^{c,d})^{-1}\}_{k\in\boldsymbol{N}}$ is the non-increasing rearrangement of $\{(\sigma(cm,dn))^{-1}\}_{m,n\in\boldsymbol{Z}^d}$, from (4.2) it follows that $\{(\sigma(cm,dn))^{-1}\} \in L_d^{p,q}(\boldsymbol{Z}^{2d})$. Clearly $1/\sigma$ is a weight. Therefore, by Proposition 4.1, we obtain that $1/\sigma \in L^{p,q}(\boldsymbol{R}^{2d})$. To prove the other implication assume that $1/\sigma \in L^{p,q}(\boldsymbol{R}^{2d})$. By Proposition 4.1 we have $\{(\sigma(am,bn))^{-1}\} \in L_d^{p,q}(\boldsymbol{Z}^{2d})$. Since $\{(\mu_k^{a,b})^{-1}\}_{k\in\boldsymbol{N}}$ is the non-increasing rearrangement of $\{(\sigma(am,bn))^{-1}\}_{m,n\in\boldsymbol{Z}^d}$, we can see that (4.2) gives $\sigma(D,x)^{-1} \in S_{p,q}$. □

EXAMPLE 4.5. Consider the situation where a τ^+-elliptic symbol $\sigma \in L^p(\boldsymbol{R}^{2d})$. Then, by Theorem 4.2, the operator $\sigma(D,x) \in S_p$. If there exists $\sigma(D,x)^s$, then it belongs to the Schatten class $S_{p/s}$. Thus, using Theorem 4.2 again, we easily see that the symbol of $\sigma(D,x)^s$ must belong to $L^{p/s}(\boldsymbol{R}^{2d})$.

EXAMPLE 4.6. Consider the function $\sigma(z) = (2+|z|^2)^{-d}\ln^{-2}(2+|z|^2)$. Then there exists $0 < \tau < 1/2d$ such that σ is a τ^+-elliptic symbol. Observe that $\sigma \notin L_s^2(\boldsymbol{R}^{2d})$ for $s > d$, where $L_s^2(\boldsymbol{R}^{2d}) = \{f;\, f(x)(1+|x|)^s \in L^2(\boldsymbol{R}^{2d})\}$. Thus, one may not use the result of Gröchenig and Heil (Theorem 1.2, [GH]) to conclude that $\sigma(D,x) \in S_1$. However, $\sigma \in L^1(\boldsymbol{R}^{2d})$, and so, by Theorem 4.2, the operator $\sigma(D,x)$ is a trace-class operator.

Yet, we need to mention that the methods of [GH] may be modified to include function spaces with logarithmic weights ([G2]).

REFERENCES

[B] R. BEALS, Characterization of pseudodifferential operators and applications, Duke Math. J. 44 (1977), 45–57.

[CR1] W. CZAJA AND Z. RZESZOTNIK, Two remarks about spectral asymptotics of pseudodifferential operators, Colloq. Math. 80 (1999), 131–145.

[CR2] W. CZAJA AND Z. RZESZOTNIK, Pseudodifferential operators and Gabor frames: spectral asymptotics, Math. Nachr. 233/234 (2002), 77–88.

[FS] H. G. FEICHTINGER AND T. STROHMER, Gabor analysis and algorithms, Appl. Numer. Harmon. Anal., Birkhäuser Boston, Boston, MA, 1998.

[F] G. B. FOLLAND, Harmonic analysis in phase space, Princeton University Press, Princeton, NJ, 1989.

[G] P. GŁOWACKI, The Weyl asymptotic formula for a class of pseudodifferential operators, Studia Math. 127 (1998), 169–190.

[G1] K. GRÖCHENIG, Foundations of time–frequency analysis, Birkhäuser Boston, Boston, MA, 2001.

[G2] K. GRÖCHENIG, personal communication.

[GH] K. GRÖCHENIG AND C. HEIL, Modulation spaces and pseudodifferential, operators Integral Equations Operator Theory 34 (1999), 439–457.

[HRT] C. HEIL, J. RAMANATHAN AND P. TOPIWALA, Singular values of compact pseudodifferential operators, J. Funct. Anal. 150 (1997), 426–452.

[H] R. HOWE, Quantum mechanics and partial differential equations, J. Funct. Anal. 38 (1980), 188–254.

[H1] L. HÖRMANDER, On the asymptotic distribution of the pseudodifferential operators in $\boldsymbol{R}^n$, Ark. Mat. 17 (1979), 297–313.

[H2] L. HÖRMANDER, The analysis of linear partial differential operators, Springer, Berlin, 1983.

[P] J. C. T. POOL, Mathematical aspects of the Weyl correspondence, J. Math. Phys. 7 (1966), 66–76.

[RT] R. ROCHBERG AND K. TACHIZAWA, Pseudodifferential operators, Gabor frames and local trigonometric bases, Gabor analysis and algorithms, 171–192, Appl. Numer. Harmon. Anal., Birkhäuser Boston, Boston, MA, 1998.

[R] C. RONDEAUX, Classes de Schatten d'opérateurs pseudo-différentiels, Ann. Sci. École Norm. Sup. (4) 17 (1984), 67–81.

[SW] W. STENGER AND A. WEINSTEIN, Methods of intermediate problems for eigenvalues, Mathematics in Science and Engineering, Vol. 89, Academic Press, New York-London, 1972.

[T] K. TACHIZAWA, The pseudodifferential operators and Wilson bases, J. Math. Pures Appl. (9) 75 (1996), 509–529.

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