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TIME-INHOMOGENEOUS STOCHASTIC PROCESSES ON THE p-ADIC NUMBER FIELD

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Abstract. The theory of stochastic differential equation on the field of p-adics is initiated by Kochubei. In this article, we will focus on a class of random walks with a certain integrability to develop the theory of stochastic analysis in a way similar to the existing theory of stochastic analysis based on the Brownian motion. In fact, for any random walk in the class, we can introduce the notion of stochastic integral with respect to the random walk and justify the existence of the solution of the stochastic differential equations based on the random walk, where the stochastic differential equations admit not only Lipschitz continuous path dependent coefficients but also continuous coefficients growing at most linearly with respect to p-adic norm. Finally, we will see an example of stochastic process which can be covered by Dirichlet space theory and obtained also by solving stochastic differential equation.

1. Introduction. The theory of p-adic numbers has become an important language to put a reasonable interpretation of several physical phenomena. In fact, many researchers proposed to investigate the theory in mathematical physics relying on the hierarchical structure (see [4], [5], [6], [11], [12]). Some classes of stochastic processes based on the hierarchical structure are dealt with by Karwowski and Vilela-Mendes [17] and by Kochubei [19], where the importance and the history of p-adic structure in mathematical physics are explored. Also, Vladimirov, Volovich and Zelenov discussed in their monograph the Brownian motion on $\boldsymbol{Q}_p$ as a counterpart of the one on the real number field ([22]).

After Evans suggested in [9] a significance of a class of $\boldsymbol{Q}_p$-valued stochastic processes including more general ones, Albeverio and Karwowski described explicitly the transition probability and detected the corresponding Dirichlet forms ([1]). In fact, by introducing a sequence $A = \{a(m)\}_{m=-\infty}^{\infty}$ satisfying

$$a(m) \geq a(m+1) \tag{1}$$

and

$$\lim_{m\to\infty} a(m) = 0\,, \quad \lim_{m\to-\infty} a(m) > 0 \ \text{ or } \ = \infty\,, \tag{2}$$

they constructed a time homogeneous symmetric Hunt process, which is associated with the regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(\boldsymbol{Q}_p; \mu)$ determined by

$$\mathcal{E}(1_{B_1}, 1_{B_2}) = -2J(B_1, B_2) = -\frac{p^{K+L-m+1}}{(p-1)}(a(m-1) - a(m))\,,$$

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where B_1 and B_2 stand for balls with radii p^K and p^L respectively, $\mathrm{dist}(B_1, B_2) = p^m$, and μ denotes the Haar measure.

Subsequently, Yasuda derived a recurrence criterion using their expression on the transition probability in [20]. As for $\boldsymbol{Q}_p$-valued spatially inhomogeneous stochastic process, Karwowski and Vilela-Mendes [17] established a family of spatially inhomogeneous processes and Albeverio, Karwowski and Zhao [2] introduced another family of spatially inhomogeneous stochastic processes. The present author introduced a family of spatially inhomogeneous stochastic processes based on a modified derivative operator in terms of Fourier transformation (see [15]). In [19], Kochubei initiated a theory of stochastic integral and stochastic differential equation based on the α-stable processes by using the associated $\boldsymbol{Q}_p$-valued Lévy system, which provides us also with time-inhomogeneous processes. In this article, we first shed light on advantages of stochastic integral with respect to a $\boldsymbol{Q}_p$-valued stochastic process in a restrictive class, and will establish another approach to the theory of stochastic differential equations which admits path dependent coefficients. We attempt extending Kochubei's framework so that it covers a stochastic differential equation with Lipschitz continuous coefficient based on various $\boldsymbol{Q}_p$-valued stochastic processes. Kochubei obtained a solution of the stochastic differential equation with Lipschitz continuous coefficient whose Lipschitz constant varies according to the width of jump of the α-stable process ([19]). However, we can solve stochastic differential equations without such an assumption on the Lipschitz continuity of the coefficient. The coverage of our formulation is completely consistent with Kochubei's stochastic integral and stochastic differential equation as long as both approaches work.

For mathematical foundations of p-adic numbers, the readers are referred to [22]. Stimulating discussions with Professor M. Takeda and Professor K. Yasuda are gratefully acknowledged. The author expresses his gratitude to the referee of this article for valuable suggestions.

2. Stochastic integral. We begin with the definition of stochastic integral based on the $\boldsymbol{Q}_p$-valued stochastic process corresponding to a sequence $A = \{a(m)\}_{m=-\infty}^{\infty}$ satisfying (1), (2) and

$$\sum_{m=-\infty}^{\infty} a(m)p^{\gamma m} < \infty \quad \text{for some given number } \gamma \geq 1\,. \tag{3}$$

More specifically, let us denote the family of all sequences with these three properties by $\mathcal{A}(\gamma)$. Then we fix a real number γ with $\gamma \geq 1$, pick out a sequence A from $\mathcal{A}(\gamma)$ and consider Albeverio and Korwowski's random walk $\{X(t)\}$ corresponding to A, which is a $\mathcal{D}([0, T] \to \boldsymbol{Q}_p)$-valued random variable, where $\mathcal{D}([0, T] \to \boldsymbol{Q}_p)$ stands for the space of all right continuous sample paths $\omega : [0, T] \to \boldsymbol{Q}_p$ with left limit at every point. The goal of this section is to establish the notion of stochastic integral with respect to the random walk $\{X(t)\}$ with $X(0) = 0$ determined by an arbitrarily given parameter sequence A of $\mathcal{A}(\gamma)$.

Choose a filtration $\{\mathcal{F}_t\}$ satisfying $\mathcal{F}_t \supset \sigma[X(s) \,|\, s \leq t]$ for any t so that $\{\mathcal{F}_t\}$ is independent of $\sigma[X(s+t) - X(t) \,|\, s > 0]$ for every $t \geq 0$. Denote by $\mathcal{S}_T$ the set of $\phi = \sum_{i=0}^{n-1} f_i 1_{[t_i, t_{i+1})}$, where $\{t_i\}_{i=0}^{n}$ is a division $0 = t_0 < t_1 < \cdots < t_n = T$ of

$[0, T]$ and each f_i is an $\{\mathcal{F}_{t_i}\}$-measurable $\boldsymbol{Q}_p$-valued random variable. Then for any element $\phi = \sum_{i=0}^{n-1} f_i 1_{[t_i, t_{i+1})} \in \mathcal{S}_T$, the stochastic integral $\int_0^t \phi(s) dX(s)$ with respect to $\{X(t)\}$ is defined by

$$\int_0^t \phi(s) dX(s) = \sum_{i=0}^{n-1} f_i (X(t_{i+1} \wedge t) - X(t_i \wedge t)) \quad \text{for} \quad 0 \le t \le T .$$

The family of random variables $\{\int_0^t \phi(s) dX(s)\}_{t \in [0,T]}$ can be regarded as an $\{\mathcal{F}_t\}$-adapted process as well as a $\mathcal{D}([0, T] \to \boldsymbol{Q}_p)$-valued random variable.

LEMMA 1. *There exists a positive constant $C_{A,\gamma}$ satisfying the following two properties*:

(i) $E[\|X(t)\|_p^\gamma] \le C_{A,\gamma} t$ *for all* $t \ge 0$,

(ii) $$E\left[\sup_{0 \le t \le u} \left\| \int_0^t \phi(s) dX(s) \right\|_p^\gamma\right] \le C_{A,\gamma} \sum_{i=0}^{n} E[\|f_i\|_p^\gamma](t_{i+1} \wedge u - (t_i \wedge u))$$

for $0 \le u \le T$.

PROOF. (i) Since the function $P_m(t) = P(X(t) \in B(0, p^m))$ has the expression

$$P_m(t) = \frac{p-1}{p} \sum_{i=0}^{\infty} p^{-i} \exp\left(- \frac{pa(m+i) - a(m+i+1)}{p-1} t \right)$$

obtained by Albeverio and Karwowski ([1]), we easily see that

$$\begin{aligned}
E[\|X(t)\|_p^\gamma] &= \sum_{m=-\infty}^{\infty} p^{\gamma m} (P_m(t) - P_{m-1}(t)) \\
&\le \frac{p-1}{p} \sum_{m=-\infty}^{\infty} p^{\gamma m} \sum_{i=0}^{\infty} p^{-i} \left(1 - \exp\left(- \frac{pa(m+i-1) - a(m+i)}{p-1} t \right) \right) \\
&\le \frac{p-1}{p} \sum_{m=-\infty}^{\infty} p^{\gamma m} \sum_{i=0}^{\infty} p^{-i} \frac{pa(m+i-1) - a(m+i)}{p-1} t \\
&\le \sum_{m=-\infty}^{\infty} p^{\gamma m} \sum_{i=0}^{\infty} p^{-i} a(m+i-1) t \\
&\le \frac{p}{p-1} \sum_{m=-\infty}^{\infty} p^{\gamma m} a(m-1) t .
\end{aligned}$$

Therefore, we can choose $p(p-1)^{-1} \sum_{m=-\infty}^{\infty} p^{\gamma m} a(m-1)$ as $C_{A,\gamma}$ for the estimate.

(ii) Since one observes that

$$\left\| \int_0^t \phi(s)dX(s) \right\|_p^\gamma = \left\| \sum_{i=0}^{n-1} f_i(X(t_{i+1} \wedge t) - X(t_i \wedge t)) \right\|_p^\gamma$$
$$\leq \max_{i=0,\dots,n} \| f_i(X(t_{i+1} \wedge t) - X(t_i \wedge t)) \|_p^\gamma$$
$$\leq \sum_{i=0}^{n-1} \| f_i \|_p^\gamma \| (X(t_{i+1} \wedge t) - X(t_i \wedge t)) \|_p^\gamma ,$$

it is not difficult to see that

$$\max_{j=0,\dots,n} \left\| \int_0^{t_j \wedge u} \phi(s)dX(s) \right\|_p^\gamma$$
$$\leq \max_{j=0,\dots,n} \sum_{i=0}^{n-1} \| f_i \|_p^\gamma \| (X(t_{i+1} \wedge t_j \wedge u) - X(t_i \wedge t_j \wedge u)) \|_p^\gamma$$
$$\leq \sum_{i=0}^{n-1} \| f_i \|_p^\gamma \| (X(t_{i+1} \wedge u) - X(t_i \wedge u)) \|_p^\gamma .$$

Accordingly, by using the constant $C_{A,\gamma}$ in (i), one sees that

$$E\left[\max_{j=0,\dots,n} \left\| \int_0^{t_j \wedge u} \phi(s)dX(s) \right\|_p^\gamma \right]$$
$$\leq E\left[\sum_{i=0}^{n-1} E[\| f_i \|_p^\gamma \| X(t_{i+1} \wedge u) - X(t_i \wedge u) \|_p^\gamma \mid \mathcal{F}_{t_i}] \right]$$
$$\leq C_{A,\gamma} E\left[\sum_{i=0}^{n-1} \| f_i \|_p^\gamma (t_{i+1} \wedge u - t_i \wedge u) \right].$$

If we take a finer division $\Delta : 0 = s_0 < s_1 < \cdots < s_m = T$ of $[0, T]$ than the original one $\Delta_\phi : 0 = t_0 < t_1 < \cdots < t_n = T$ of the function $\phi = \sum_{i=0}^{n-1} f_i 1_{[t_i, t_{i+1})}$, then ϕ admits another expression $\phi = \sum_{i=0}^{m-1} g_j 1_{[s_j, s_{j+1})}$, and it turns out that

$$E\left[\max_{j=0,\dots,m} \left\| \int_0^{s_j \wedge u} \phi(s)dX(s) \right\|_p^\gamma \right] \leq C_{A,\gamma} E\left[\sum_{j=0}^{m-1} \| g_j \|_p^\gamma (s_{j+1} \wedge u - s_j \wedge u) \right]$$
$$= C_{A,\gamma} E\left[\sum_{i=0}^{n-1} \| f_i \|_p^\gamma (t_{i+1} \wedge u - t_i \wedge u) \right].$$

By passing to the limit as $\|\Delta\| \to 0$, one can verify

$$E\left[\sup_{0 \leq t \leq u} \left\| \int_0^t \phi(s)dX(s) \right\|_p^\gamma \right] \leq C_{A,\gamma} E\left[\sum_{i=0}^{n-1} \| f_i \|_p^\gamma (t_{i+1} \wedge u - t_i \wedge u) \right]. \qquad \square$$

Denote the set of $\boldsymbol{Q}_p$-valued random variables X satisfying $E[\|X\|_p^\gamma] < \infty$ by L^γ, and denote the set of $\{\mathcal{F}_t\}$-adapted $\boldsymbol{Q}_p$-valued processes regarded as continuous maps from $[0, T]$ to L^γ by $\mathcal{C}([0, T] \to L^\gamma)$. For any element $\phi \in \mathcal{C}([0, T] \to L^\gamma)$, there exists a sequence $\{\phi_n\} \subset \mathcal{S}_T$ such that $\lim_{n\to\infty} \sup_{0\le t\le T} E[\|\phi_m(t) - \phi(t)\|_p^\gamma] = 0$. Then one can derive from Lemma 1 (ii) that

$$\lim_{n,m\to\infty} E\left[\sup_{0\le t\le T} \left\| \int_0^t \phi_n(s)dX(s) - \int_0^t \phi_m(s)dX(s)\right\|_p^\gamma\right] = 0 .$$

Therefore, the stochastic integral $\int_0^t \phi(s)dX(s)$ of ϕ with respect to $\{X(t)\}$ can be defined as a unique $\boldsymbol{Q}_p$-valued process $\{Y(t)\}$ satisfying

$$\lim_{n\to\infty} E\left[\sup_{0\le t\le T} \left\| Y(t) - \int_0^t \phi_n(s)dX(s)\right\|_p^\gamma\right] = 0 .$$

Since we already know that $\int_0^\cdot \phi_n(s)dX(s)$ is a $\mathcal{D}([0, T] \to \boldsymbol{Q}_p)$-valued random variable, we immediately see that $\int_0^\cdot \phi(s)dX(s)$ is a $\mathcal{D}([0, T] \to \boldsymbol{Q}_p)$-valued variable as well.

PROPOSITION 1. *The stochastic integral has the following properties*:

(i) $$E\left[\sup_{u\le t\le v} \left\| \int_u^t \phi(s)dX(s)\right\|_p^\gamma\right] \le C_{A,\gamma} E\left[\int_u^v \|\phi(s)\|_p^\gamma ds\right],$$

(ii) $$\int_0^t (\phi(s) + \psi(s))dX(s) = \int_0^t \phi(s)dX(s) + \int_0^t \psi(s)dX(s),$$

(iii) $$\int_0^u \phi(s)dX(s) = \int_0^v \phi(s)dX(s) + \int_v^u \phi(s)dX(s) \quad \textit{for all } v, u \textit{ with } u \ge v,$$

(iv) $\int_0^t \phi(s)dX(s)$ *is a* $\boldsymbol{Q}_p$*-valued process in* $\mathcal{C}([0, T] \to L^\gamma)$.

PROOF. (i) Similarly to the proof of Lemma 1 (ii), for any $\phi \in \mathcal{S}_T$, one can verify

$$E\left[\sup_{v\le t\le u} \left\| \int_v^t \phi(s)dX(s)\right\|_p^\gamma\right] \le C_{A,\gamma} \sum_{i=0}^n E[\|f_i\|_p^\gamma]((t_{i+1} \wedge u) \vee v - ((t_i \wedge u) \vee v),$$

for all $0 \le u, v \le T$ with $u \ge v$. From this one can conclude that

$$E\left[\sup_{v\le t\le u} \left\| \int_v^u \phi(s)dX(s)\right\|_p^\gamma\right] \le C_{A,\gamma} E\left[\int_v^u \|\phi(s)\|_p^\gamma ds\right]$$

for all $u, v \in [0, T]$ with $u \ge v$.

(ii) and (iii) follow immediately from the definition of the stochastic integral.

(vi) Since the second term of the right-hand side in (iii) converges to zero in L^γ as $u \to v$, (vi) has been proved. □

For two $\boldsymbol{Q}_p$-valued stochastic processes $\{X_1(t)\}$ and $\{X_2(t)\}$, a fundamental calculation shows that

$$
\begin{aligned}
X_1(t)X_2(t) - X_1(0)X_2(0)
&= \sum_{i=0}^{n-1}\left(X_1\left(\frac{(i+1)t}{n}\right)X_2\left(\frac{(i+1)t}{n}\right) - X_1\left(\frac{it}{n}\right)X_2\left(\frac{it}{n}\right)\right)\\
&= \sum_{i=0}^{n-1} X_1\left(\frac{it}{n}\right)\left(X_2\left(\frac{(i+1)t}{n}\right) - X_2\left(\frac{it}{n}\right)\right)\\
&\quad + \sum_{i=0}^{n-1} X_2\left(\frac{it}{n}\right)\left(X_1\left(\frac{(i+1)t}{n}\right) - X_1\left(\frac{it}{n}\right)\right)\\
&\quad + \sum_{i=0}^{n-1}\left(X_1\left(\frac{(i+1)t}{n}\right) - X_1\left(\frac{it}{n}\right)\right)\left(X_2\left(\frac{(i+1)t}{n}\right) - X_2\left(\frac{it}{n}\right)\right).
\end{aligned}
$$

In particular, when X_1 and X_2 are random walks corresponding to some sequences of $\mathcal{A}(\gamma)$ for a fixed $\gamma \geq 1$, the first and the second terms converge to $\int_0^t X_1(s)dX_2(s)$ and to $\int_0^t X_2(s)dX_1(s)$, respectively. As a consequence of this fact, we obtain the following two examples:

EXAMPLE 1. For the random walk $\{X(t)\}$ corresponding to $A \in \mathcal{A}(\gamma)$ with $\gamma \geq 2$, there exists an element $\{Z(t)\}$ of $\mathcal{C}([0,T] \to L^{\gamma/2})$ satisfying

$$X(t)^2 - X(0)^2 = 2\int_0^t X(s)dX(s) + Z(t)\,.$$

$\{Z(t)\}$ is characterized also as a unique element of $\mathcal{C}([0,T] \to L^{\gamma})$ enjoying

$$\lim_{n\to\infty} E\left[\left\| Z(t) - \sum_{i=0}^{n-1}\left(X\left(\frac{(i+1)t}{n}\right) - X\left(\frac{it}{n}\right)\right)^2\right\|_p^{\gamma}\right] = 0\,.$$

EXAMPLE 2. For independent two random walks $\{X_1(t)\}$ and $\{X_2(t)\}$ corresponding to A_1 and $A_2 \in \mathcal{A}(\gamma)$ $(\gamma \geq 1)$ respectively, we have the following formula:

$$X_1(t)X_2(t) - X_1(0)X_2(0) = \int_0^t X_1(s)dX_2(s) + \int_0^t X_2(s)dX_1(s)\,,$$

since

$$
\begin{aligned}
&E\left[\left\|\sum_{i=0}^{n-1}\left(X_1\left(\frac{(i+1)t}{n}\right)-X_1\left(\frac{it}{n}\right)\right)\left(X_2\left(\frac{(i+1)t}{n}\right)-X_2\left(\frac{it}{n}\right)\right)\right\|_p^\gamma\right] \\
&\quad\leq E\left[\max_i\left\|\left(X_1\left(\frac{(i+1)t}{n}\right)-X_1\left(\frac{it}{n}\right)\right)\left(X_2\left(\frac{(i+1)t}{n}\right)-X_2\left(\frac{it}{n}\right)\right)\right\|_p^\gamma\right] \\
&\quad\leq E\left[\sum_{i=0}^{n-1}\left\|\left(X_1\left(\frac{(i+1)t}{n}\right)-X_1\left(\frac{it}{n}\right)\right)\left(X_2\left(\frac{(i+1)t}{n}\right)-X_2\left(\frac{it}{n}\right)\right)\right\|_p^\gamma\right] \\
&\quad=\sum_{i=0}^{n-1}E\left[\left\|X_1\left(\frac{(i+1)t}{n}\right)-X_1\left(\frac{it}{n}\right)\right\|_p^\gamma\right]E\left[\left\|X_2\left(\frac{(i+1)t}{n}\right)-X_2\left(\frac{it}{n}\right)\right\|_p^\gamma\right] \\
&\quad\leq C_{A_1,\gamma}C_{A_2,\gamma}\sum_{i=0}^{n-1}\left(\frac{(i+1)t}{n}-\frac{it}{n}\right)^2 \\
&\quad= C_{A_1,\gamma}C_{A_2,\gamma}\frac{t^2}{n}\to 0 \quad (n\to\infty)\,.
\end{aligned}
$$

3. Stochastic differential equation with path dependent coefficient. We consider a function $\sigma(t, X)$ defined on $[0, T]\times\mathcal{C}([0, T]\to L^\gamma)$ for some $\gamma\geq 1$ enjoying the following properties:

(4) for each $X\in\mathcal{C}([0, T]\to L^\gamma)$, $\sigma(\cdot, X)$ defines an element of $\mathcal{C}([0, T]\to L^\gamma)$,

(5) for each $s\in[0, T]$, any element $X\in\mathcal{C}([0, T]\to L^\gamma)$ gives an $\{\mathcal{F}_s\}$-measurable random variable $\sigma(s, X)$ depending only on the family of random variables $\{X(u)\,|\,u\in[0, s]\}$,

(6) there exists a positive constant C_T satisfying

$$
E[\|\sigma(t, X)-\sigma(t, X')\|_p^\gamma]\leq C_T E\left[\sup_{0\leq u\leq t}\|X(u)-X'(u)\|_p^\gamma\right]
$$

for any $X, X'\in\mathcal{C}([0, T]\to L^\gamma)$ and any $t\in[0, T]$.

For an element σ enjoying these properties (4), (5) and (6) and a random walk $\{X(t)\}$ corresponding to some A of $\mathcal{A}(\gamma)$, if an element $\{Y(t)\}\in\mathcal{C}([0, T]\to L^\gamma)$ satisfies the stochastic integral equation

$$
Y(t)=x+\int_0^t\sigma(s, Y)dX(s)\,,\quad 0\leq t\leq T\,,
$$

for some starting point $x\in\boldsymbol{Q}_p$, then $\{Y(t)\}$ is called a solution of the stochastic differential equation

$$
\begin{cases} dY(t)=\sigma(t, Y)dX(t)\,,\\ Y(0)=x\,.\end{cases}
$$

THEOREM 1. *If $\{X(t)\}$ is a random walk corresponding to some sequence A of $\mathcal{A}(\gamma)$ with $\gamma \geq 1$, then the stochastic differential equation*

$$\begin{cases} dY(t) = \sigma(t, Y)dX(t)\,, \\ Y(0) = x\,, \end{cases}$$

has a unique solution $\{Y(t)\}$ for every starting point $x \in \boldsymbol{Q}_p$.

PROOF. Define a sequence $\{Y^{(n)}(t)\}_{n=0}^{\infty}$ of $\mathcal{C}([0,T] \to L^{\gamma})$ inductively by

$$\begin{cases} Y^{(0)}(t) = x, \\ Y^{(n)}(t) = x + \displaystyle\int_0^t \sigma(s, Y^{(n-1)})dX(s)\,, \quad n = 1, 2, \dots\,. \end{cases}$$

From Proposition 1 (i), it follows that

$$E\Big[\sup_{0\leq u\leq t} \|Y^{(1)}(u) - x\|_p^{\gamma}\Big] \leq KC_{A,\gamma}t \quad \text{for} \quad K = \sup_{0\leq t\leq T} E[\|\sigma(t,x)\|_p^{\gamma}]\,.$$

In general, by assuming

$$E\Big[\sup_{0\leq u\leq t} \|Y^{(n)}(u) - Y^{(n-1)}(u)\|_p^{\gamma}\Big] \leq KC_{A,\gamma}^{n}C_T^{n-1}\frac{t^n}{n!}\,,$$

one can derive

$$E\Big[\sup_{0\leq u\leq t} \|Y^{(n+1)}(u) - Y^{(n)}(u)\|_p^{\gamma}\Big] \leq KC_{A,\gamma}^{n+1}C_T^{n}\frac{t^{n+1}}{(n+1)!}\,,$$

since

$$\begin{aligned} E\Big[\sup_{0\leq u\leq t} &\|Y^{(n+1)}(u) - Y^{(n)}(u)\|_p^{\gamma}\Big] \\ &\leq E\Big[\sup_{0\leq u\leq t} \Big\| \int_0^u (\sigma(s, Y^{(n)}) - \sigma(s, Y^{(n-1)}))dX(s)\Big\|_p^{\gamma}\Big] \\ &\leq C_{A,\gamma}E\Big[\int_0^t \|\sigma(s, Y^{(n)}) - \sigma(s, Y^{(n-1)})\|_p^{\gamma}ds\Big] \\ &= C_{A,\gamma}\int_0^t E[\|\sigma(s, Y^{(n)}) - \sigma(s, Y^{(n-1)})\|_p^{\gamma}]ds \\ &\leq C_{A,\gamma}C_T\int_0^t E\Big[\sup_{0\leq u\leq s} \|(Y^{(n)}(u) - Y^{(n-1)}(u))\|_p^{\gamma}\Big]ds \\ &\leq KC_{A,\gamma}^{n+1}C_T^{n}\frac{t^{n+1}}{(n+1)!}\,. \end{aligned}$$

Therefore, combining the estimate

$$P\Big[\sup_{0\leq u\leq T} \|Y^{(n+1)}(t) - Y^{(n)}(t)\|_p^{\gamma} > \frac{1}{p^n}\Big] \leq K\frac{(C_{A,\gamma}T)^{n+1}(C_T p)^n}{(n+1)!}$$

with Borel-Cantelli lemma, one can see that there exists an $\{\mathcal{F}_t\}$-adapted process $\{Y(t)\}$ satisfying

$$P\left[\lim_{n\to\infty}\sup_{0\le t\le T}\|Y^{(n)}(t)-Y(t)\|_p=0\right]=1\,.$$

On the other hand, since one observes that

$$E\left[\sup_{0\le t\le T}\|Y^{(n)}(t)-Y^{(m)}(t)\|_p^{\gamma}\right]\le K\sum_{k=n}^{m}\frac{(C_{A,\gamma}T)^{k+1}(C_T)^k}{(k+1)!}\quad (m>n)\,,$$

Fatou's Lemma shows that

$$E\left[\sup_{0\le t\le T}\|Y^{(n)}(t)-Y(t)\|_p^{\gamma}\right]\le K\sum_{k=n}^{\infty}\frac{(C_{A,\gamma}T)^{k+1}(C_T)^k}{(k+1)!}\,.$$

Therefore, the sequence $\{Y^{(n)}(t)\}_{n=1}^{\infty}$ converges to $\{Y(t)\}$ in L^{γ} uniformly in $t\in[0,T]$ as n goes to ∞. Combining this inequality with the assumption (6), one can derive from Proposition 1 (i) that

$$\begin{aligned}E&\left[\sup_{0\le u\le t}\left\|\int_0^u(\sigma(s,Y)-\sigma(s,Y^{(n)}))dX(s)\right\|_p^{\gamma}\right]\\&\le C_{A,\gamma}C_TE\left[\int_0^t\sup_{0\le u\le s}\left\|Y(s)-Y^{(n)}(s)\right\|_p^{\gamma}ds\right]\to 0\quad (n\to\infty)\,.\end{aligned}$$

Consequently, from

$$Y^{(n)}(t)=x+\int_0^t\sigma(s,Y^{(n-1)})dX(s)\quad (n=1,2,\dots)$$

we can derive

$$Y(t)=x+\int_0^t\sigma(s,Y)dX(s)\,.$$

This also ensures that $\{Y(t)\}$ is an element of $\mathcal{C}([0,T]\to L^{\gamma})$ and a $\mathcal{D}([0,T]\to \boldsymbol{Q}_p)$-valued random variable as well.

On the other hand, if another element $\{Z(t)\}\in\mathcal{C}([0,T]\to L^{\gamma})$ satisfies

$$\begin{cases}dZ(t)=\sigma(t,Z)dX(t)\,,\\ Z(0)=x\,,\end{cases}$$

then we can immediately derive the equality $\{Z(t)\}=\{Y(t)\}$ as the elements of $\mathcal{C}([0,T]\to L^{\gamma})$ from the following estimates:

$$\begin{aligned}E&\left[\sup_{0\le u\le t}\|Y(u)-Z(u)\|_p^{\gamma}\right]\\&=E\left[\sup_{0\le u\le t}\left\|\int_0^u(\sigma(s,Y)-\sigma(s,Z))dX(s)\right\|_p^{\gamma}\right]\\&\le C_{A,\gamma}C_TE\left[\int_0^t\sup_{0\le u\le s}\|Y(u)-Z(u)\|_p^{\gamma}ds\right].\end{aligned}$$

□

4. Stochastic differential equation with continuous coefficient of linear growth. We consider a function $\sigma_0(t,x)$ on $[0,T]\times \boldsymbol{Q}_p$ enjoying the following properties:

(7) each $t\in[0,T]$ defines a continuous map $\sigma_0(t,\cdot):\boldsymbol{Q}_p\to\boldsymbol{Q}_p$,

(8) there exists a sequence $\{\sigma_n(t,\cdot)\}$ of coefficients defined on $[0,T]\times\boldsymbol{Q}_p$ so that every coefficient admits a positive constant $C_{n,T}$ satisfying

$$\sup_{t\in[0,T]}\|\sigma_n(t,x)-\sigma_n(t,x')\|_p\le C_{n,T}\|x-x'\|_p\quad\text{for all } x,x'\in\boldsymbol{Q}_p$$

and

$$\lim_{n\to\infty}\sup_{\substack{t\in[0,T]\\ x\in B(0,p^N)}}\|\sigma_n(t,x)-\sigma_0(t,x)\|_p=0\quad\text{for each positive integer } N,$$

(9) there exists a non-negative function ψ defined on $[0,\infty)$ satisfying $\lim_{\xi\to0}\psi(\xi)=0$ and $E[\|\sigma_n(t,X(t))-\sigma_n(t',X(t'))\|_p^2]\le\psi(E[\|X(t)-X(t')\|_p^2])$ for any $X\in\mathcal{C}([0,T]\to L^2)$ and each $n=0,1,2,\dots$,

(10) there exists a positive constant K_0 satisfying

$$\sup_{t\in[0,T]}\|\sigma_n(t,x)\|_p\le K_0(1+\|x\|_p)$$

for any $x\in\boldsymbol{Q}_p$ and $n=0,1,2,\dots$.

By virtue of Theorem 1, given a random walk $\{X(t)\}$ corresponding to A of $\mathcal{A}(2)$, the assumptions (8) and (9) enable us to define a sequence $\{Y^{(n)}(t)\}_{n=1}^\infty$ of $\mathcal{C}([0,T]\to L^2)$ by

$$Y^{(n)}(t)=x+\int_0^t\sigma_n(s,Y^{(n)}(s))dX(s)\,,\quad n=1,2,\dots.$$

LEMMA 2. *If a random walk $\{X(t)\}$ corresponds to some A of $\mathcal{A}(1)\cap\mathcal{A}(2)$, then there exist positive constants K_1, K_2 and K_3, depending only on the starting point x and on the sequence A, which satisfy the following properties*:

(i) $E\left[\sup_{0\le u\le t}\|Y^{(n)}(u)\|_p^2\right]\le K_2\exp(K_1t)\,,\quad n=1,2,\dots,$

(ii) $E\left[\sup_{s\le u\le t}\|Y^{(n)}(u)-Y^{(n)}(s)\|_p^2\right]\le 2C_{A,2}K_0^2(1+K_2\exp(K_1T))(t-s)\,,$ $n=1,2,\dots,$

(iii) $E[\|Y^{(n)}(t'+h')-Y^{(n)}(t')\|_p\|Y^{(n)}(t)-Y^{(n)}(t-h)\|_p]\le K_3h'h^{1/2}$, $n=1,2,\dots$, *for any $t,t'\in[0,T]$ with $t'\ge t$ and for any positive numbers h,h' enjoying $t-h,\ t'+h'\in[0,T]$.*

PROOF. (i) Since the assumption (10) together with Proposition 1 (i) implies that

$$E\Big[\sup_{0\le u\le t}\|Y^{(n)}(u)-x\|_p^2\Big]=E\Big[\sup_{0\le u\le t}\Big\|\int_0^u\sigma_n(s,Y^{(n)}(s))dX(s)\Big\|_p^2\Big]$$
$$\le C_{A,2}K_0^2E\Big[\int_0^t(1+\|Y^{(n)}(s)\|_p)^2ds\Big]$$
$$\le K_1E\Big[\int_0^t(1+\|Y^{(n)}(s)-x\|_p^2)ds\Big]$$
$$\le K_1E\Big[\int_0^t(1+\sup_{0\le u\le s}\|Y^{(n)}(u)-x\|_p^2)ds\Big]$$

for some constant K_1, one easily sees

$$E\Big[\sup_{0\le u\le t}\|Y^{(n)}(u)-x\|_p^2\Big]\le\exp(K_1t)\,.$$

From this one can immediately derive the estimate (i).

(ii) By applying the inequality in Proposition 1 (i) to the right-hand side of the identity $Y^{(n)}(u)-Y^{(n)}(s)=\int_s^u\sigma_n(v,Y^{(n)}(v))dX(v)$, we can utilize (i) to obtain estimate (ii).

(iii) Since the stochastic integral is defined as the limit of the sequence of stochastic integrals with integrands in $\mathcal{S}_T$, it suffices to show that for any $\{\mathcal{F}_{t'}\}$-measurable random variable f with $E[\|f\|_p^2]\le 2K_0^2(1+K_2\exp(K_1T))$, there exists a positive constant K_3 satisfying

$$E[\|f(X(t'+h')-X(t'))\|_p\|Y^{(n)}(t)-Y^{(n)}(t-h)\|_p]\le K_3h'h^{1/2}\,,$$

which easily follows from the following inequalities:

$$\begin{aligned}
&E[\|f(X(t'+h')-X(t'))\|_p\|Y^{(n)}(t)-Y^{(n)}(t-h)\|_p]\\
&\quad=E[E[\|f(X(t'+h')-X(t'))\|_p\|Y^{(n)}(t)-Y^{(n)}(t-h)\|_p\,|\,\mathcal{F}_t]]\\
&\quad=E[E[\|(X(t'+h')-X(t'))\|_p]E[\|f\|_p\|Y^{(n)}(t)-Y^{(n)}(t-h)\|_p\,|\,\mathcal{F}_t]]\\
&\quad\le C_{A,1}h'E[\|f\|_p\|Y^{(n)}(t)-Y^{(n)}(t-h)\|_p]\\
&\quad\le C_{A,1}h'E[\|f\|_p^2]^{1/2}E[\|Y^{(n)}(t)-Y^{(n)}(t-h)\|_p^2]^{1/2}\\
&\quad\le C_{A,1}h'\times 2C_{A,2}^{1/2}K_0^2(1+K_2\exp(K_1T))h^{1/2}\,.
\end{aligned}$$

□

If an element $\{Y(t)\}\in\mathcal{C}([0,T]\to L^2)$ satisfies the stochastic integral equation

$$Y(t)=x+\int_0^t\sigma(s,Y(s))dX(s)\,,\quad 0\le t\le T\,,$$

for some point $x\in\boldsymbol{Q}_p$ and some random walk $\{X(t)\}$ corresponding to A, then the stochastic differential equation

$$\begin{cases}dY(t)=\sigma(t,Y(t))dX(t)\,,\\ Y(0)=x\,,\end{cases}$$

is said to have a solution with respect to a random walk corresponding to A, and x is called a starting point.

THEOREM 2. *If a random walk* $\{X(t)\}$ *corresponds to some* A *of* $\mathcal{A}(1) \cap \mathcal{A}(2)$, *then the stochastic differential equation*

$$\begin{cases} dY(t) = \sigma_0(t, Y(t))dX(t)\,, \\ Y(0) = x\,, \end{cases}$$

has a solution with respect to a random walk corresponding to A *for every starting point* $x \in \boldsymbol{Q}_p$.

PROOF. By combining Lemma 2 with a result in Ethier and Kurtz [8], it is known that the family $\{Y^{(n)}(t \wedge T), X(t \wedge T)\}_{t\in[0,\infty)}$ of $\boldsymbol{Q}_p \times \boldsymbol{Q}_p$-valued stochastic processes is tight, where $\boldsymbol{Q}_p \times \boldsymbol{Q}_p$ is considered as a metric space with the norm $|(x, y)| = \|x\|_p + \|y\|_p$ $(x, y \in \boldsymbol{Q}_p)$. Therefore, there exists a family $\{\bar{Y}^{(n)}(t), \bar{X}^{(n)}(t)\}_{n=0}^{\infty}$ of $\boldsymbol{Q}_p \times \boldsymbol{Q}_p$-valued processes which has the following two properties:

(11) for each $n = 1, 2, \ldots$, $\{\bar{Y}^{(n)}(t), \bar{X}^{(n)}(t)\}$ has the same probability law as $\{Y^{(n)}(t \wedge T), X(t \wedge T)\}_{t\in[0,\infty)}$,

(12) $\{\bar{Y}^{(n)}(t), \bar{X}^{(n)}(t)\}$ converges to $\{\bar{Y}^{(0)}(t), \bar{X}^{(0)}(t)\}$ in the space $\mathcal{D}([0,\infty) \to \boldsymbol{Q}_p \times \boldsymbol{Q}_p)$ as $n \to \infty$.

These properties ensure that $\{\bar{X}^{(0)}(t)\}$ and $\{X(t)\}$ have the same probability law and that

$$E\left[\sup_{0\le u\le t} \|\bar{Y}^{(0)}(u)\|_p^2\right] \le K_2 \exp(K_1 t) \quad \text{ for all } t \in [0, T]\,.$$

Therefore, Lemma 2 (ii) and assumption (9) show that, for any $\varepsilon > 0$, there exists a division $0 = t_0 < t_1 < \cdots < t_k = T$ of $[0, T]$ such that

$$E\left[\left\| \sum_{i=0}^{k-1} \sigma_n(t_i, \bar{Y}^{(n)}(t_i))(\bar{X}^{(n)}(t_{i+1} \wedge t) - \bar{X}^{(n)}(t_i \wedge t)) \right.\right.$$
$$\left.\left. - \int_0^t \sigma_n(s, \bar{Y}^{(n)}(s))d\bar{X}^{(n)}(s) \right\|_p^2\right]^{1/2} \le \varepsilon$$

for $n = 0, 1, 2, \ldots$. This implies the finiteness of

$$\sup_n E\left[\left\| \sum_{i=0}^{k-1} \sigma_0(t_i, \bar{Y}^{(n)}(t_i))(\bar{X}^{(n)}(t_{i+1} \wedge t) - \bar{X}^{(n)}(t_i \wedge t)) \right\|_p^2\right],$$

which yields the uniform integrability of the family of the random variables $\{\| \sum_{i=0}^{k-1} \sigma_0(t_i, \bar{Y}^{(n)}(t_i))(\bar{X}^{(n)}(t_{i+1} \wedge t) - \bar{X}^{(n)}(t_i \wedge t))\|_p\}_{n=0}^{\infty}$. Accordingly, in the inequality

$$\begin{aligned}
E\Bigg[\Bigg\| & \sum_{i=0}^{k-1} \sigma_n(t_i, \bar{Y}^{(n)}(t_i))(\bar{X}^{(n)}(t_{i+1} \wedge t) - \bar{X}^{(n)}(t_i \wedge t)) \\
&\quad - \sum_{i=0}^{k-1} \sigma_0(t_i, \bar{Y}^{(0)}(t_i))(\bar{X}^{(0)}(t_{i+1} \wedge t) - \bar{X}^{(0)}(t_i \wedge t))\Bigg\|_p\Bigg] \\
&\le E\Bigg[\Bigg\| \sum_{i=0}^{k-1} \sigma_n(t_i, \bar{Y}^{(n)}(t_i))(\bar{X}^{(n)}(t_{i+1} \wedge t) - \bar{X}^{(n)}(t_i \wedge t)) \\
&\quad - \sum_{i=0}^{k-1} \sigma_0(t_i, \bar{Y}^{(n)}(t_i))(\bar{X}^{(n)}(t_{i+1} \wedge t) - \bar{X}^{(n)}(t_i \wedge t))\Bigg\|_p\Bigg] \\
&\quad + E\Bigg[\Bigg\| \sum_{i=0}^{k-1} \sigma_0(t_i, \bar{Y}^{(n)}(t_i))(\bar{X}^{(n)}(t_{i+1} \wedge t) - \bar{X}^{(n)}(t_i \wedge t)) \\
&\quad - \sum_{i=0}^{k-1} \sigma_0(t_i, \bar{Y}^{(0)}(t_i))(\bar{X}^{(0)}(t_{i+1} \wedge t) - \bar{X}^{(0)}(t_i \wedge t))\Bigg\|_p\Bigg],
\end{aligned}$$

the property (12) shows that the second expectation of the right-hand side does not exceed any given positive number ε as long as n is sufficiently large, since the family of real-valued random variables $\{\| \sum_{i=0}^{k-1} \sigma_0(t_i, \bar{Y}^{(n)}(t_i))(\bar{X}^{(n)}(t_{i+1} \wedge t) - \bar{X}^{(n)}(t_i \wedge t))\|_p\}_{n=0}^{\infty}$ is uniformly integrable.

On the other hand, the first expectation of the right-hand side is dominated by the sum of the following expectations:

$$\begin{aligned}
E\Bigg[\Bigg\| & \sum_{i=0}^{k-1} (\sigma_n(t_i, \bar{Y}^{(n)}(t_i)) - \sigma_0(t_i, \bar{Y}^{(n)}(t_i))) \\
&\quad \times (\bar{X}^{(n)}(t_{i+1} \wedge t) - \bar{X}^{(n)}(t_i \wedge t))\Bigg\|_p 1_{\{\sup_{0 \le u \le t} \|\bar{Y}^{(n)}(u)\| \le p^N\}}\Bigg]
\end{aligned}$$

and

$$\begin{aligned}
E\Bigg[\Bigg\| & \sum_{i=0}^{k-1} (\sigma_n(t_i, \bar{Y}^{(n)}(t_i)) - \sigma_0(t_i, \bar{Y}^{(n)}(t_i))) \\
&\quad \times (\bar{X}^{(n)}(t_{i+1} \wedge t) - \bar{X}^{(n)}(t_i \wedge t))\Bigg\|_p 1_{\{\sup_{0 \le u \le t} \|\bar{Y}^{(n)}(u)\| > p^N\}}\Bigg].
\end{aligned}$$

The former expectation is dominated by

$$\sup_{x \in B(0, p^N)} \|\sigma_n(t, x) - \sigma_0(t, x)\|_p E\Bigg[\Bigg\| \sum_{i=0}^{k-1} (\bar{X}^{(n)}(t_{i+1} \wedge t) - \bar{X}^{(n)}(t_i \wedge t))\Bigg\|_p\Bigg].$$

From Lemma 1 (ii), we see that this admits the upper bound $C_{A,1}T \times \sup_{t\in[0,T],x\in B(0,p^N)} \|\sigma_n(t,x)-\sigma_0(t,x)\|_p$. Therefore, from the assumption (8), we see that this goes to zero as $n \to \infty$. By applying Schwarz' inequality to the latter expectation, we obtain its dominant

$$E\left[\left\| \sum_{i=0}^{k-1}(\sigma_n(t_i,\bar{Y}^{(n)}(t_i)) - \sigma_0(t_i,\bar{Y}^{(n)}(t_i))) \times (\bar{X}^{(n)}(t_{i+1}\wedge t) - \bar{X}^{(n)}(t_i\wedge t))\right\|_p^2\right]^{1/2} P\left(\sup_{0\le u\le t}\|\bar{Y}^{(n)}(u)\| > p^N\right)^{1/2}.$$

Here, Lemma 1 (ii) implies the following estimate on the expectation:

$$\begin{aligned}E\left[\left\| \sum_{i=0}^{k-1}(\sigma_n(t_i,\bar{Y}^{(n)}(t_i)) - \sigma_0(t_i,\bar{Y}^{(n)}(t_i)))(\bar{X}^{(n)}(t_{i+1}\wedge t) - \bar{X}^{(n)}(t_i\wedge t))\right\|_p^2\right] \\ \le C_{A,2}E\left[\sum_{i=0}^{k-1}\|\sigma_n(t_i,\bar{Y}^{(n)}(t_i)) - \sigma_0(t_i,\bar{Y}^{(n)}(t_i))\|_p^2((t_{i+1}\wedge t)-(t_i\wedge t))\right]\end{aligned}$$

for $n = 1, 2, \ldots$. Combining Lemma 2 (i) with the assumption (10), we know that the supremum of the expectations of the right-hand side remains finite. Again from Lemma 2 (i), we can derive the following estimate:

$$\begin{aligned}P\left(\sup_{0\le u\le t}\|\bar{Y}^{(n)}(u)\|_p > p^N\right) &= P\left(\sup_{0\le u\le t}\|Y^{(n)}(u)\|_p > p^N\right) \\ &\le E\left[\sup_{0\le u\le t}\|Y^{(n)}(u)\|_p^2\right]/p^{2N} \le K_2\exp(K_1 t)/p^{2N}.\end{aligned}$$

Since the right-hand side tends to zero as $N \to \infty$, the expectation

$$E\left[\left\| \sum_{i=0}^{k-1}(\sigma_n(t_i,\bar{Y}^{(n)}(t_i)) - \sigma_0(t_i,\bar{Y}^{(n)}(t_i))) \times (\bar{X}^{(n)}(t_{i+1}\wedge t) - \bar{X}^{(n)}(t_i\wedge t))\right\|_p 1_{\{\sup_{0\le u\le t}\|\bar{Y}^{(n)}(u)\|>p^N\}}\right]$$

converges to zero as $N \to \infty$. Accordingly, for any positive number ε, as long as n is sufficiently large, we can verify that

$$\begin{aligned}
&E\Big[\Big\|\int_0^t \sigma_n(s,\bar{Y}^{(n)}(s))d\bar{X}^{(n)}(s) - \int_0^t \sigma_0(s,\bar{Y}^{(0)}(s))d\bar{X}^{(0)}(s)\Big\|_p\Big] \\
&\quad \le E\Big[\Big\|\int_0^t \sigma_n(s,\bar{Y}^{(n)}(s))d\bar{X}^{(n)}(s) \\
&\qquad\quad - \sum_{i=0}^{k-1}\sigma_n(t_i,\bar{Y}^{(n)}(t_i))(\bar{X}^{(n)}(t_{i+1}\wedge t) - \bar{X}^{(n)}(t_i\wedge t))\Big\|_p\Big] \\
&\qquad + E\Big[\Big\|\sum_{i=0}^{k-1}\sigma_n(t_i,\bar{Y}^{(n)}(t_i))(\bar{X}^{(n)}(t_{i+1}\wedge t) - \bar{X}^{(n)}(t_i\wedge t)) \\
&\qquad\quad - \sum_{i=0}^{k-1}\sigma_0(t_i,\bar{Y}^{(0)}(t_i))(\bar{X}^{(0)}(t_{i+1}\wedge t) - \bar{X}^{(0)}(t_i\wedge t))\Big\|_p\Big] \\
&\qquad + E\Big[\Big\|\sum_{i=0}^{k-1}\sigma_0(t_i,\bar{Y}^{(0)}(t_i))(\bar{X}^{(0)}(t_{i+1}\wedge t) - \bar{X}^{(0)}(t_i\wedge t)) \\
&\qquad\quad - \int_0^t \sigma_0(s,\bar{Y}^{(0)}(s))d\bar{X}^{(0)}(s)\Big\|_p\Big] \le 3\varepsilon\,.
\end{aligned}$$

Consequently, by passing to the limit as $n \to \infty$, from the equality

$$\bar{Y}^{(n)}(t) = x + \int_0^t \sigma_n(s,\bar{Y}^{(n)}(s))d\bar{X}^{(n)}(s)\,,$$

we can derive

$$\bar{Y}^{(0)}(t) = x + \int_0^t \sigma_0(s,\bar{Y}^{(0)}(s))d\bar{X}^{(0)}(s)\,.$$

As a result, we conclude that the stochastic differential equation

$$\begin{cases} dY(t) = \sigma_0(t,Y(t))dX(t)\,, \\ Y(0) = x\,, \end{cases}$$

has a solution with respect to a random walk corresponding to A of $\mathcal{A}(1)\cap\mathcal{A}(2)$ for every starting point $x \in \boldsymbol{Q}_p$. □

5. Stochastic analysis with respect to random walks in a larger family. In this section, we will obtain wider perspectives of p-adic stochastic analysis so that it covers the theory of stochastic differential equations based on the α-stable processes. For this purpose, we consider a random walk corresponding to a sequence $A = \{a(m)\}$ satisfying (1), (2) and

(13) $$\sum_{m=-\infty}^{0} a(m)p^{\gamma m} < \infty \quad \text{for a given real number } \gamma \ge 1\,.$$

Denote the family of all sequences with these three properties by $\bar{\mathcal{A}}(\gamma)$ and define sequences $A(M) = \{a(M; m)\}$ of $\mathcal{A}(\gamma)$ by

$$a(M; m) = \begin{cases} a(m) & \text{if } m < M\,, \\ 0 & \text{if } m \geq M\,. \end{cases}$$

PROPOSITION 2. *For any random walk $\{X(t)\}$ corresponding to A of $\bar{\mathcal{A}}(\gamma)$, we have the following*:

(i) *The $\boldsymbol{Q}_p$-valued process $\{X(M; t)\}$ defined by $X(M; t) = \int_{\boldsymbol{Q}_p} \int_0^t f_M(z) N(ds, dz)$ is a random walk corresponding to $A(M)$, where N stands for the Poisson random measure of $\{X(t)\}$ and f_M denotes the $\boldsymbol{Q}_p$-valued function defined by*

$$f_M(z) = \begin{cases} p^m z & \textit{if} \quad \|z\|_p = p^{m+M}, \quad m \geq 0\,, \\ z & \textit{otherwise}\,. \end{cases}$$

(ii) *There exists a sequence $\{\Omega(M; T)\}_{M=0}^{\infty}$ of events satisfying $\lim_{M\to\infty} P(\Omega(M; T)) = 1$ and*

$$\int_0^t \phi(s) dX(M; s) = \int_0^t \phi(s) dX(M + k; s)$$

for any $t \in [0, T]$ and $k = 1, 2, \ldots$ a.s. on $\Omega(M; T)$ for any ϕ of $\mathcal{C}([0, T] \to L^{\gamma})$.

(iii) *For any sequence $\{\phi(M; t)\}_{M=0}^{\infty} \subset \mathcal{C}([0, T] \to L^{\gamma})$ enjoying $\phi(M; t) = \phi(M + k; t)$ for any $t \in [0, T]$ and $k = 1, 2, \ldots$ a.s. on $\Omega(M; T)$,*

$$\int_0^t \phi(M; s) dX(M; s) = \int_0^t \phi(M + k; s) dX(M + k; s)$$

for any $t \in [0, T]$ and $k = 1, 2, \ldots$ a.s. on $\Omega(M; T)$.

PROOF. (i) Since we have, for any ball $B(x, p^k) \subset B(0, p^M) \setminus B(0, p^{M-1})$,

$$\begin{aligned} \lim_{t\to 0} \frac{P(X(M; t) \in B(x, p^k))}{t} &= \lim_{t\to 0} \frac{P(X(t) \in \bigcup_{m=0}^{\infty} B(p^{-m}x, p^{k+m}))}{t} \\ &= \sum_{m=0}^{\infty} \frac{p^{k+m-(M+m)+1}(a(M + m - 1) - a(M + m))}{p - 1} \\ &= \frac{p^{k-M+1} a(M - 1)}{p - 1}\,, \end{aligned}$$

it follows that $\{X(M; t)\}$ has the same infinitesimal generator as the random walk determined by $A(M)$.

(ii) Let $\Omega(M; T)$ denote the event {the sample path has no jump larger than p^M before T}. Then, Yasuda's result shows that $P(\Omega(M; T)) = \exp(-a(M)T) \to 1$ as $M \to \infty$ (see [20]). The assertion immediately follows from the fact that

$$\int_0^t \phi(s) dX(M; s) = \int_0^t \phi(s) dX(M + k; s) \quad k = 1, 2, \ldots \quad \text{a.s. on} \quad \Omega(M; T)$$

for every $\phi \in \mathcal{S}_T$.

(iii) There exist two sequences $\{\phi_n(M;s)\}_{n=0}^{\infty}$ and $\{\phi_n(M+k;s)\}_{n=0}^{\infty}$ of $\mathcal{S}_T$ satisfying $\lim_{n\to\infty}\sup_{0\le t\le T}E[\|\phi_n(M;t)-\phi(M;t)\|_p^{\gamma}]=0$ and $\lim_{n\to\infty}\sup_{0\le t\le T}E[\|\phi_n(M+k;t)-\phi(M+k;t)\|_p^{\gamma}]=0$ respectively so that $\phi_n(M;s)=\phi_n(M+k;s),\ 0\le s\le T,$ on $\Omega(M;T)$. Therefore, we can derive the assertion from an argument similar to that for the proof of (ii). □

Consequently, the stochastic integral $\{\int_0^t \phi(s)dX(s)\}_{t\in[0,T]}$ with respect to an arbitrarily given random walk $\{X(t)\}$ determined by A of $\bar{\mathcal{A}}(\gamma)$ can be defined as a unique $\{\mathcal{F}_t\}$-adapted process $\{Y(t)\}$ with continuity in probability satisfying

$$Y(t)=\int_0^t \phi(M;s)dX(M;s) \quad \text{a.s. on } \Omega(M;T) \quad \text{for all } M=0,1,2,\dots,$$

as long as the $\{\mathcal{F}_t\}$-adapted process $\{\phi(t)\}$ associates a sequence $\{\phi(M;t)\}_{M=0}^{\infty}\subset\mathcal{C}([0,T]\to L^{\gamma})$ enjoying

$$\phi(t)=\phi(M;t) \quad \text{for any } t\in[0,T] \quad \text{a.s. on } \Omega(M;T),\ M=0,1,\dots.$$

REMARK. We can derive similar conclusions as in Example 1 and Example 2. In fact, for the random walk $\{X(t)\}$ corresponding to $A\in\bar{\mathcal{A}}(\gamma)$ with $\gamma\ge 2$, there exists a unique $\boldsymbol{Q}_p$-valued process $\{Z(t)\}$ with continuity in probability satisfying

$$X(t)^2-X(0)^2=2\int_0^t X(s)dX(s)+Z(t),$$

where $Z(t)$ is characterized as the limit of $\{\sum_{i=0}^{n-1}(X((i+1)t/n)-X(it/n))^2\}_{n=1}^{\infty}$ in probability. Furthermore, for independent two random walks $\{X_1(t)\}$ and $\{X_2(t)\}$ corresponding to A_1 and $A_2\in\bar{\mathcal{A}}(\gamma)$ $(\gamma\ge 1)$ respectively, we have the following formula:

$$X_1(t)X_2(t)-X_1(0)X_2(0)=\int_0^t X_1(s)dX_2(s)+\int_0^t X_2(s)dX_1(s).$$

If an $\{\mathcal{F}_t\}$-adapted process $\{Y(t)\}$ with continuity in probability satisfies the stochastic integral equation

$$Y(t)=x+\int_0^t \sigma(s,Y(s))dX(s), \quad 0\le t\le T,$$

for a starting point $x\in\boldsymbol{Q}_p$ and for a random walk $\{X(t)\}$ corresponding to A of $\bar{\mathcal{A}}(\gamma)$, then $\{Y(t)\}$ is called a solution of stochastic differential equation

$$\begin{cases} dY(t)=\sigma(t,Y(t))dX(t), \\ Y(0)=x. \end{cases}$$

THEOREM 3. *If a $\boldsymbol{Q}_p$-valued function $\sigma(t,x)$ defined on $[0,T]\times\boldsymbol{Q}_p$ admits a positive constant C_T satisfying*

$$\|\sigma(t,x)-\sigma(t,x')\|_p\le C_T\|x-x'\|_p$$

for any $x,x'\in\boldsymbol{Q}_p$ and $t\in[0,T]$ and if each X of $\mathcal{C}([0,T]\to L^{\gamma}))$ gives an element $\sigma(\cdot,X(\cdot))$ of $\mathcal{C}([0,T]\to L^{\gamma})$, then for any random walk $\{X(t)\}$ corresponding to A of $\bar{\mathcal{A}}(\gamma)$

there exists a unique solution $\{Y(t)\}$ *of the stochastic differential equation*

$$\begin{cases} dY(t) = \sigma(t, Y(t))dX(t)\,, \\ Y(0) = x\,, \end{cases}$$

for every starting point $x \in \boldsymbol{Q}_p$. *In particular, if the coefficient is given as* $\sigma(x)$ *independently of the time variable* t, *then the solution* $\{Y(t)\}$ *is characterized as a unique Markov process with the infinitesimal generator described as*:

$$\lim_{t\to 0} \frac{1}{t}(E_x[u(Y_t)] - u(x))$$
$$= \begin{cases} \displaystyle\int_{\boldsymbol{Q}_p} (u(y) - u(x))c(\log_p \|(y-x)/\sigma(x)\|_p)\|\sigma(x)\|_p^{-1}\mu(dy) & \text{if } \sigma(x) \neq 0\,, \\ 0 & \text{if } \sigma(x) = 0\,, \end{cases}$$

for any locally constant function u *with compact support*, *where* $c(m) = (p-1)^{-1}p^{1-m}(a(m-1) - a(m))$.

PROOF. By virtue of Theorem 1, we obtain a solution $\{Y(M;t)\}$ of the stochastic integral equation

$$Y(M;t) = x + \int_0^t \sigma(s, Y(M;s))dX(M;s)$$

for every $M = 0, 1, \dots$.

It is not difficult to see that

$$Y(M;t) = Y(M+k;t) \quad \text{for } t \in [0,T] \text{ and } k = 1, 2, \dots \quad \text{a.s. on } \Omega(M;T)\,,$$

and that the solution of the stochastic integral equation

$$Y(t) = x + \int_0^t \sigma(s, Y(s))dX(s)$$

is obtained as a unique $\{\mathcal{F}_t\}$-adapted process $\{Y(t)\}$ satisfying

$$Y(t) = Y(M;t) \quad \text{for } t \in [0,T] \quad \text{a.s. on } \Omega(M;T) \quad \text{for } M = 0, 1, \dots.$$

Then clearly $\{Y(t)\}$ is a $\mathcal{D}([0,T] \to \boldsymbol{Q}_p)$-valued random variable.

On the other hand, if the stochastic integral equation admits another solution $\{Z(t)\}$, then

$$\begin{aligned} E[\|Z(t\wedge\tau(M)) - Y(t\wedge\tau(M))\|_p^\gamma] &= E\left[\left\| \int_0^{t\wedge\tau(M)} (\sigma(s, Z(s)) - \sigma(s, Y(s)))dX(s) \right\|_p^\gamma\right] \\ &\leq C_{A(M),\gamma} C_T^\gamma \int_0^t E[\|Z(s\wedge\tau(M)) - Y(s\wedge\tau(M))\|_p^\gamma]ds\,, \end{aligned}$$

for any $t \in [0,T]$ and for any $\tau(M) = \inf\{t > 0 \mid \|X(t)\|_p > p^M\}$. Hence we have $Z(t) = Y(t)$ a.s. $(t < \tau(M))$ for every $M = 0, 1, \dots$, which shows the uniqueness of the solution of the stochastic differential equation.

It is not difficult to show that $\{Y(t)\}$ is a Markov process by the standard procedure as in Kochubei [19].

The proof of Theorem 1 also yields the following estimate with some positive constant $C_{\sigma,M}$:

$$E[\|Y(M;s)-x\|_p^\gamma] \le C_{\sigma,M}t\,.$$

On the other hand, for the $\{\mathcal{F}_t\}$-adapted process

$$\hat{Y}(M;t) = x + \int_0^t \sigma(x)dX(M;s) = x + \sigma(x)X(M;t)\,,$$

one sees that $P(\hat{Y}(M;t)\in B(y,p^k)) = P(X(M;t)\in B((y-x/\sigma(x),\, p^k\|\sigma(x)\|_p^{-1})$ for any $y\in \boldsymbol{Q}_p$, $k\in\boldsymbol{Z}$ and x with $\sigma(x)\neq 0$. Hence one deduces that

$$\lim_{t\to 0}\frac{1}{t}(E_x[u(\hat{Y}(M;t))]-u(x)) = \begin{cases} \displaystyle\int_{\mathbf{Q}_p}(u(y)-u(x))c_M(\log_p\|(y-x)/\sigma(x)\|_p)\|\sigma(x)\|_p^{-1}\mu(dy) & \text{if } \sigma(x)\neq 0\,,\\ 0 & \text{if } \sigma(x)=0\,,\end{cases}$$

where $c_M(m) = (p-1)^{-1}p^{1-m}(a(M;m-1)-a(M;m))$. One sees also that

$$\begin{aligned} E[|u(Y(M;t)) - u(\hat{Y}(M;t))|] &\le C(u)E[\|Y(M;t)-\hat{Y}(M;t)\|_p^\gamma]\\ &= C(u)E\left[\left\|\int_0^t(\sigma(Y(M;s))-\sigma(x))dX(M;s)\right\|_p^\gamma\right]\\ &\le C(u)C_{A(M),\gamma}\int_0^t E[\|(\sigma(Y(M;s))-\sigma(x))\|_p^\gamma]ds\\ &\le C(u)C_{A(M),\gamma}C_T^\gamma\int_0^t E[\|Y(M;s)-x\|_p^\gamma]ds\\ &\le C(u)C_{A(M),\gamma}C_T^\gamma C_{\sigma,M}t^2\,,\end{aligned}$$

by using the constant $C(u) = \sup_{x,y\in\boldsymbol{Q}_p}|u(x)-u(y)|/\|x-y\|_p^\gamma$ of u. These two facts then imply

$$\lim_{t\to 0}\frac{1}{t}(E_x[u(Y(M;t))]-u(x)) = \begin{cases} \displaystyle\int_{\mathbf{Q}_p}(u(y)-u(x))c_M(\log_p\|(y-x)/\sigma(x)\|_p)\|\sigma(x)\|_p^{-1}\mu(dy) & \text{if } \sigma(x)\neq 0\,,\\ 0 & \text{if } \sigma(x)=0\,.\end{cases}$$

Accordingly, from the following estimates

$$\begin{aligned}
&\left| \frac{1}{t} E_x[u(Y(t)) - u(x)] - \frac{1}{t} E_x[(u(Y(M;t)) - u(x))\right| \\
&\quad = \left| \frac{1}{t} E_x[(u(Y(t)) - u(x))1_{\Omega(M;t)}] + \frac{1}{t} E_x[(u(Y(t)) - u(x))1_{\Omega(M;t)^c}] \right. \\
&\qquad \left. - \frac{1}{t} E_x[(u(Y(M;t)) - u(x))1_{\Omega(M;t)}] - \frac{1}{t} E_x[(u(Y(M;t)) - u(x))1_{\Omega(M;t)^c}] \right| \\
&\quad \le 4 \sup |u| \frac{1}{t} P(\Omega(M;t)^c) \\
&\quad = 4 \sup |u| \frac{1}{t} (1 - \exp(-a(M)t) \\
&\quad \le 4 \sup |u| a(M)\,,
\end{aligned}$$

one can derive that $|\lim_{t\to 0}(1/t)E_x[u(Y(t)) - u(x)]| \le 4 \sup |u| a(M)$ at every point x with $\sigma(x) = 0$. Similarly, one sees that, for some positive constant K_4 independent of $a(M)$,

$$\begin{aligned}
&\left| \lim_{t\to 0} \frac{1}{t} E_x[u(Y(t)) - u(x)] \right. \\
&\qquad \left. - \int_{\boldsymbol{Q}_p} (u(y) - u(x)) c(\log_p \|(y - x)/\sigma(x)\|_p) \|\sigma(x)\|_p^{-1} \mu(dy) \right| \\
&\quad \le \left| \lim_{t\to 0} \frac{1}{t} E_x[u(Y(t)) - u(x)] - \lim_{t\to 0} \frac{1}{t} E_x[(u(Y(M;t)) - u(x))] \right| \\
&\qquad + \left| \int_{\boldsymbol{Q}_p} (u(y) - u(x))(c_M(\log_p \|(y - x)/\sigma(x)\|_p) \right. \\
&\qquad\qquad \left. - c(\log_p \|(y - x)/\sigma(x)\|_p)) \|\sigma(x)\|_p^{-1} \mu(dy) \right| \\
&\quad \le K_4 \sup |u| a(M)
\end{aligned}$$

at every point $x \in \boldsymbol{Q}_p$ with $\sigma(x) \ne 0$. Since $a(M)$ tends to zero as $M \to \infty$, the assertion is legitimized. □

EXAMPLE 3. If a Lipschitz continuous function σ satisfies $\inf \|\sigma\|_p > 0$ and $\sup \|\sigma\|_p < \infty$, for the random walk $\{X(t)\}$ characterized by $A = \{cp^{-\alpha m}\}_{m=-\infty}^{\infty}$ $(\alpha > 0)$ with some positive constant c, the probability law of the solution $\{Y(t)\}$ of the stochastic differential equation

$$\begin{cases} dY(t) = \sigma(Y(t))dX(t)\,, \\ Y(0) = x\,, \end{cases}$$

coincides with the one of the Hunt process corresponding to the smallest closed extension in $L^2(\|\sigma\|_p^{1-\alpha}\mu)$ of the pre-Dirichlet form

$$\mathcal{E}(u, v) = \int_{(\boldsymbol{Q}_p \times \boldsymbol{Q}_p)\setminus \Delta} (u(x) - u(y))(v(x) - v(y)) J(dx, dy)$$

with the symmetric measure J on $(\boldsymbol{Q}_p \times \boldsymbol{Q}_p) \setminus \Delta$ $(\Delta = \{(x, x) \mid x \in \boldsymbol{Q}_p\})$ given by

$$J(dx, dy) = k_{p,c,\alpha} \|y - x\|_p^{-(\alpha+1)} \mu(dx)\mu(dy)\,,$$

where $k_{p,c,\alpha}$ stands for some positive constant depending only on p, c and α and u, v denote locally constant functions with compact supports.

6. $\boldsymbol{Q}_p$-valued martingale. In this section, we assume that some increasing filtration $\{\mathcal{F}_t\}$ is given on a probability space $(\Omega, \mathcal{F}, P)$. For a square integrable random variable X, if there exists a unique element Y of $L^2(\Omega, \mathcal{F}_t, P)$ satisfying $E[\|X - Y\|_p^2] = \inf_{Z \in L^2(\Omega, \mathcal{F}_t, P)} E[\|X - Z\|_p^2]$, we denote Y by $E[X \mid \mathcal{F}_t]$ and call the random variable the conditional expectation of X given by $\mathcal{F}_t$. Also, if there exists a unique element a of $\boldsymbol{Q}_p$ satisfying $E[\|X - a\|_p^2] = \inf_{z \in \boldsymbol{Q}_p} E[\|X - z\|_p^2]$, we denote a by $E[X]$ and call the p-adic number the expectation of X.

PROPOSITION 3. (i) *If the probability distribution μ_X of a square integrable random variable X satisfies $\mu_X(B(a, p^k)) > \mu_X(B(b, p^k))$ for any $a, b \in \boldsymbol{Q}_p$ with $\|a\|_p < \|b\|_p$ and any integer k with $p^k < \|a - b\|_p$, then $E[X] = 0$.*

(ii) *If X is an $\mathcal{F}_t$-measurable square integrable random variable, then $E[X \mid \mathcal{F}_t] = X$.*

(iii) *If a square integrable random variable X with expectation is independent of $\mathcal{F}_t$, then $E[X \mid \mathcal{F}_t] = E[X]$.*

PROOF. (i) For any element a of $\boldsymbol{Q}_p$ with $\|a\|_p = p^{m_a}$, we have

$$\begin{aligned}
&E[\|X - a\|_p^2] - E[\|X\|_p^2]\\
&\quad = \sum_{m=m_a+1}^{\infty} p^{2m}\mu_X(S(0, p^m)) + p^{2m_a}\mu_X(B(0, p^{m_a-1}))\\
&\qquad + p^{2m_a}\mu_X(S(0, p^{m_a}) \setminus B(a, p^{m_a-1}))\\
&\qquad + \sum_{m=-\infty}^{m_a-1} p^{2m}\mu_X(S(a, p^m)) - \sum_{m=-\infty}^{\infty} p^{2m}\mu_X(S(0, p^m))\\
&\quad = p^{2m_a}(\mu_X(B(0, p^{m_a-1})) - \mu_X(B(a, p^{m_a-1})))\\
&\qquad + \sum_{m=-\infty}^{m_a-1} p^{2m}\mu_X(S(a, p^m)) - \sum_{m=-\infty}^{m_a-1} p^{2m}\mu_X(S(0, p^m))\\
&\quad = (\mu_X(B(0, p^{m_a-1})) - \mu_X(B(a, p^{m_a-1})))\\
&\qquad \times \left(p^{2m_a} - \sum_{m=-\infty}^{m_a-1} p^{2m}\frac{\mu_X(S(0, p^m)) - \mu_X(S(a, p^m))}{\mu_X(B(0, p^{m_a-1})) - \mu_X(B(a, p^{m_a-1}))}\right)\\
&\quad > 0\,.
\end{aligned}$$

(ii) follows directly from the definition.

(iii) For any $\mathcal{F}_t$-measurable random variable Z in L^2, denote its probability distribution by μ_Z. Then we observe that if $\mu_Z(\{0\}) < 1$,

$$E[\|X-(E[X]+Z)\|_p^2] = \int_{\mathbf{Q}_p} E[\|X-(E[X]+z)\|_p^2]\mu_Z(dz)$$
$$> \int_{\mathbf{Q}_p} E[\|X-E[X]\|_p^2]\mu_Z(dz)\,.$$

Therefore, it follows that the infimum is attained if and only if $\mu_Z(\{0\}) = 1$. □

A $\boldsymbol{Q}_p$-valued $\{\mathcal{F}_t\}$-adapted square integrable process $\{M_t\}$ is called an $\{\mathcal{F}_t\}$-martingale if $E[M_t \mid \mathcal{F}_s] = M_s$ for any $t > s$.

THEOREM 4. *A-random walk $\{X(t)\}$ characterized by a parameter sequence $A \in \mathcal{A}(2)$ is an $\{\mathcal{F}_t\}$-martingale.*

PROOF. Denoting $b_i(t) = p^{-i}\exp(-(p-1)^{-1}(pa(i)-a(i+1))t)$, the following description on $P_0(X(t) \in B(0, p^m))$ is obtained ([1]):

$$P_m(t) = \frac{p-1}{p}\sum_{i=0}^{\infty} p^{-i}\exp\left(-\frac{pa(m+i)-a(m+i+1)}{p-1}t\right)$$
$$= \frac{p-1}{p}p^m\sum_{i=m}^{\infty} b_i(t)\,.$$

On the other hand, for any element a with $\|a\|_p = p^{m_a} > p^m$, one can see that

$$P_0(X(t) \in B(a, p^m)) = \frac{p}{p-1}\frac{1}{p^{m_a-m}}(P_{m_a}(t) - P_{m_a-1}(t))$$
$$= \frac{p^m}{p^{m_a}}\left(p^{m_a}\sum_{i=m_a}^{\infty} b_i(t) - p^{m_a-1}\sum_{i=m_a-1}^{\infty} b_i(t)\right)$$
$$= p^m\left(\sum_{i=m_a}^{\infty} b_i(t) - \frac{1}{p}\sum_{i=m_a-1}^{\infty} b_i(t)\right)$$
$$\le \frac{p-1}{p}p^m\sum_{i=m_a}^{\infty} b_i(t) < P_m(t)\,.$$

Consequently, $E[X(t)] = 0$ and $E[X(t)\,|\,\mathcal{F}_s] = E[X(t)-X(s)+X(s)\,|\,\mathcal{F}_s] = E[X(t)-X(s)\,|\,\mathcal{F}_s] + X(s) = E[X(t)-X(s)] + X(s) = X(s)$. □

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