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ON THE STRUCTURE OF THE SPACE OF METRICS DEFINED ON A GIVEN SET

Abstract

This paper is a continuation of the authors' earlier paper [4]. Denote by $\mathcal{M}$ the metric space of all metrics on a given nonvoid set X with the sup-metric. In this paper some subsets of $\mathcal{M}$ are investigated, namely subsets consisting of all metrics $d \in \mathcal{M}$ for which (X, d) possesses prescribed topological (metric) properties.

1. Introduction

Let X be a given nonvoid set. Denote by $\mathcal{M} = \mathcal{M}(X)$ the set of all metrics on X endowed with the metric

$$d^*(d, d') = \min\{1, \sup_{x,y \in X} |d(x, y) - d'(x, y)|\} \text{ for } d,\ d' \in \mathcal{M}.$$

It is the purpose of this paper to investigate the structure of the metric space $(\mathcal{M}, d^*)$ and examine the properties of the following sets:

$$\begin{aligned}
\mathcal{U} &= \mathcal{U}(X) = \{d \in \mathcal{M}; (X, d) \text{ is a complete metric space }\}\\
\mathcal{S} &= \mathcal{S}(X) = \{d \in \mathcal{M}; (X, d) \text{ is a separable metric space }\}\\
\mathcal{K} &= \mathcal{K}(X) = \{d \in \mathcal{M}; (X, d) \text{ is a compact metric space }\}\\
\mathcal{C} &= \mathcal{C}(X) = \{d \in \mathcal{M}; (X, d) \text{ is a connected metric space}\}, \text{ and}
\end{aligned}$$

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$\mathcal{T} = \{t_a \in \mathcal{M}; t_a(x,y) = a > 0 \text{ for } x \neq y \in X, \text{ and } t_a(x,x) = 0 \text{ for } x \in X\}^1$

For each $a > 0$ put

$$\mathcal{H}_a^* = \{d \in \mathcal{M}; \forall_{x,\,y \in X} d(x,y) < a\},$$

$$\mathcal{H}_a = \{d \in \mathcal{M}; \forall_{\substack{x,\,y\in X\\ x\neq y}} d(x,y) \geq a\} \text{ and } \mathcal{H} = \cup_{a>0}\mathcal{H}_a.$$

We have ([4], Lemma 2)

Lemma A *The set $\mathcal{H}$ is an open and dense subset of $\mathcal{M}$.*

Throughout this paper we will use the notation from [4]. In what follows suppose that $|X| \geq 2$.

2. Main Results

The equivalence of metrics determines an equivalence relation $\sim$ on $\mathcal{M}$. The symbol $\mathcal{M}|_\sim$ stands for the set of all equivalence classes generated by $\sim$.

Let $\mathcal{O}_0$ be the class from $\mathcal{M}|_\sim$ whose elements d fulfill the following property: the sequence $x_k \in X$ ($k \in \mathbb{N}$) converges with respect to d if and only if $\{x_k\}_{k=1}^\infty$ is almost stationary (i.e. $d(x_k, x) \to 0$ as $k \to \infty$ for some $x \in X$ implies that $x_k = x$ for all but at most finitely many k).

The class $\mathcal{O}_0$ contains all trivial metrics, but there are other metrics, too. For instance if $X = (0, +\infty)$ and $\varrho(x,y) = \max\{x,y\}$ for $x \neq y$, and $\varrho(x,x) = 0$, then $\varrho \in \mathcal{O}_0 \setminus \mathcal{T}$ (cf. [2]).

Theorem 1 *The sets $\mathcal{O}_0$ and $\mathcal{U}$ are residual subsets of the 2nd category in $(\mathcal{M}, d^*)$.*

PROOF. It is an easy consequence of Lemma 2 and Theorem 3 in [4]. □

Remark 1 *It is worth saying that the spaces $(\mathcal{O}_0, d^*)$ and $(\mathcal{U}, d^*)$ are not complete, since the sequence $t_{\frac{1}{k}} \in \mathcal{T} \subset \mathcal{O}_0 \subset \mathcal{U}$ ($k \in \mathbb{N}$)is fundamental and has no limit in $\mathcal{O}_0$ and $\mathcal{U}$, respectively.*

Theorem 2 *Each of the sets $\mathcal{S}, \mathcal{K}$ and $\mathcal{C}$ can be represented as a union of classes from $\mathcal{M}|_\sim$.*

PROOF. The assertion follows from the fact that the properties of separability, compactness and connectedness, respectively are topological properties. Thus if $d \in \mathcal{S}$ ($d \in \mathcal{K}, d \in \mathcal{C}$) is in the class $\mathcal{O}$, then $\mathcal{O} \subset \mathcal{S}$ ($\mathcal{O} \subset \mathcal{K}, \mathcal{O} \subset \mathcal{C}$). □

The analogous theorem for $\mathcal{U}$ does not hold in general. Actually we have

[1]the elements of $\mathcal{T}$ are the so-called trivial metrics.

Theorem 3 *(i) If X is a finite set, then $\mathcal{U} = \mathcal{M} = \mathcal{O}_0$.*

(ii) If X is an infinite set, then $\mathcal{O}_0 \cap \mathcal{U} \neq \emptyset \neq \mathcal{O}_0 \cap (\mathcal{M} \setminus \mathcal{U})$.

PROOF. Case (i) is trivial.
(ii) Let X be an infinite set. Then there exists a one-to-one sequence $x_k \in X$ $(k \in \mathbb{N})$. Put $X' = X \setminus \{x_1, x_2, \ldots, x_k, \ldots\}$ and define the metric ϱ on X as follows: $\varrho(x,x) = 0$ for $x \in X, \varrho(x,y) = 1$ if $x \neq y$ and at least one of x, y belongs to X', further $\varrho(x_i, x_j) = \max\{\frac{1}{i}, \frac{1}{j}\}$ for $i \neq j, i, j \in \mathbb{N}$. We prove that $\varrho \in \mathcal{O}_0 \cap (\mathcal{M} \setminus \mathcal{U})$.

Let $\varrho(y_k, y) \to 0$ as $k \to \infty$ $(y_k, y \in X, k \in \mathbb{N})$. Then $\varrho(y_k, y) = 1$ if $y \in X', y \neq y_k$ and $\varrho(y_k, y) \geq \frac{1}{m}$ if $y = x_m, y_k \neq x_m$ $(k \in \mathbb{N})$. Thus $y_k = y$ for every $k \geq k_0$ $(k_0 \in \mathbb{N})$, which implies that $\varrho \in \mathcal{O}_0$. Further the sequence $\{x_k\}_{k=1}^{\infty}$ is evidently fundamental in (X, ϱ), but does not converge because it is not almost stationary. Consequently $\varrho \in \mathcal{M} \setminus \mathcal{U}$.

On the other hand the trivial metric $t_1 \in \mathcal{O}_0 \cap \mathcal{U}$. □

Lemma 4 *Every class $\mathcal{O} \in \mathcal{M}|_{\sim}$ is a dense in itself subset of $\mathcal{M}$. Moreover each point of $\mathcal{O}$ is its point of condensation.*

PROOF. Let $\varepsilon > 0$. Let $\mathcal{O} \in \mathcal{M}|_{\sim}, d \in \mathcal{O}$. It suffices to consider the metrics $d_a = d + a \cdot \min\{1, d\}$ for $0 < a < \varepsilon$, and notice that $d_a \in \mathcal{O} \cap K(d, \varepsilon)$ for all $0 < a < \varepsilon$. □

Theorem 5 *Each of the sets $\mathcal{S}$, $\mathcal{K}$, $\mathcal{C}$ and $\mathcal{U}$ is a dense in itself subset of $(\mathcal{M}, d^*)$.*

PROOF. The union of an arbitrary system of dense in itself sets is dense in itself (see [1], p.46). Accordingly for $\mathcal{S}$, $\mathcal{K}$ and $\mathcal{C}$ the assertion follows from Theorem 3 and Lemma 4. In view of Theorem 3(ii) we have to deal with $\mathcal{U}$ separately. Let $d \in \mathcal{U}, \varepsilon > 0$ and $0 < a < \varepsilon$. Put $d_a(x,y) = d(x,y) + a$, for $x, y \in X$, $x \neq y$ and $d_a(x,x) = 0$ for $x \in X$. Then $d_a \in \mathcal{U} \cap K(d, \varepsilon)$ for each $0 < a < \varepsilon$. □

Theorem 6 *The set $\mathcal{S}$ is closed in $(\mathcal{M}, d^*)$.*

PROOF. Suppose that d belongs to the closure of $\mathcal{S}$ in $(\mathcal{M}, d^*)$. Then there exists a sequence $d_n \in \mathcal{S}$ $(n \in \mathbb{N})$ such that $\lim_{n \to \infty} d^*(d_n, d) = 0$. There is a countable set $M_n \subset X$ dense in (X, d_n) $(n \in \mathbb{N})$. Put $M = \cup_{n=1}^{\infty} M_n$. Then M is countable. To prove M is dense in (X, d) let $x_0 \in X$ and $0 < \varepsilon < 1$. Since $d^*(d_n, d) \to 0$ $(n \to \infty)$, there exists $n_0 \in \mathbb{N}$ such that $|d_{n_0}(x,y) - d(x,y)| < \frac{\varepsilon}{2}$ for every $x, y \in X$ whence $d(x,y) < d_{n_0}(x,y) + \frac{\varepsilon}{2}$ $(x, y \in X)$. Since M_{n_0} is dense in (X, d_{n_0}), there is $y_0 \in M_{n_0} \subset M$ for which $d_{n_0}(x_0, y_0) < \frac{\varepsilon}{2}$. Consequently we get $d(x_0, y_0) < d_{n_0}(x_0, y_0) + \frac{\varepsilon}{2} < \varepsilon$. □

Remark 2 *In view of Theorems 5 and 6 it turns out that $\mathcal{S}$ is a perfect subset of $(\mathcal{M}, d^*)$.*

Theorem 7 *We have*

(i) if $2 \le |X| \le \aleph_0$, then $\mathcal{S}$ is of the 2nd category in $\mathcal{M}$, and

(ii) if $|X| > \aleph_0$, then $\mathcal{S}$ is nowhere dense in $\mathcal{M}$.

PROOF.

(i) If $2 \le |X| \le \aleph_0$, then $\mathcal{S} = \mathcal{M}$ and Theorem 3 in [4] yields the desired result.

(ii) If $|X| > \mathbf{c}$, then $\mathcal{S} = \emptyset$ (see [1], p.140). Therefore it suffices to consider the case $\aleph_0 < |X| \le \mathbf{c}$. The trivial metric t_1 is evidently in $\mathcal{O}_0 \setminus \mathcal{S}$. Furthermore, according to Theorem 2 $d \sim t_1$ implies $d \notin \mathcal{S}$. Thus $\mathcal{O}_0 \subset \mathcal{M} \setminus \mathcal{S}$. The assertion follows from Theorems 1 and 6. □

Theorem 8 *We have*

(i) if $2 \le |X| < \aleph_0$, then $\mathcal{K}$ is of the 2nd category in $\mathcal{M}$,

(ii) if $|X| \ge \aleph_0$, then $\mathcal{K}$ is nowhere dense in $\mathcal{M}$.

PROOF.

(i) In this case we have $\mathcal{K} = \mathcal{M}$.

(ii) Let $|X| \ge \aleph_0$. Then similar to the proof of Theorem 7(ii) we can show that $\mathcal{O}_0 \subset \mathcal{M} \setminus \mathcal{K}$. Thus $\mathcal{K} \subset \mathcal{M} \setminus \mathcal{O}_0 \subset \mathcal{M} \setminus \mathcal{H}$, Hence Lemma A applies.

□

Theorem 9 *The set $\mathcal{K}$ is closed in $(\mathcal{U}, d^*)$.*

PROOF. If X is a finite set, then $\mathcal{K} = \mathcal{U}$. So we can suppose that $|X| \ge \aleph_0$. Let $d_n \in \mathcal{K}$ $(n \in \mathbb{N}), d \in \mathcal{U}$ and $d^*(d_n, d) \to 0$ as $n \to \infty$. Assume that $d \in \mathcal{U} \setminus \mathcal{K}$. Then (X, d) is not totally bounded. Hence for some $1 > \varepsilon_0 > 0$ X has a countable ε_0-discrete subset, i.e. there exists a sequence $x_n \in X$ $(n \in \mathbb{N})$ such that

$$d(x_k, x_l) \ge \varepsilon_0, \text{ for all } k, l \in \mathbb{N}, k \ne l. \tag{1}$$

Let $n \in \mathbb{N}$ be fixed. The metric space (X, d_n) is compact. So $\{x_k\}_{k=1}^{\infty}$ has a convergent subsequence $\{x_{k_j}\}_{j=1}^{\infty}$ in (X, d_n). Then from (1) we have $|d(x_{k_i}, x_{k_j}) - d_n(x_{k_i}, x_{k_j})| \ge \varepsilon_0 - d_n(x_{k_i}, x_{k_j})$, for $i, j \in \mathbb{N}, i \ne j$, So

$$\sup_{i,j \in \mathbb{N}, i \ne j} |d(x_{k_i}, x_{k_j}) - d_n(x_{k_i}, x_{k_j})| \ge \varepsilon_0.$$

This implies that $d^*(d, d_n) \geq \varepsilon_0 > 0$ for every $n \in \mathbb{N}$, which is a contradiction. □

Remark 3 *It is not true in general that $\mathcal{K}$ is closed in $(\mathcal{M}, d^*)$. To see this, let $X = \{x_1, \ldots, x_i, \ldots\}$ be a countable set. Define metrics d, d_n $(n \in \mathbb{N})$ as follows: $d(x,x) = 0$ for all $x \in X$ and $d(x_i, x_j) = \max\{\frac{1}{i}, \frac{1}{j}\}$ for all $i, j \in \mathbb{N}, i \neq j$. Further for each $n \in \mathbb{N}$ put $d_n(x,x) = 0$ for $x \in X$ and for $i, j \in \mathbb{N}, i \neq j$*

$$d_n(x_i, x_j) = \begin{cases} d(x_i, x_j), & \text{if } \min\{i,j\} \neq n \\ \min\{\frac{1}{i}, \frac{1}{j}\}, & \text{if } \min\{i,j\} = n. \end{cases}$$

Then $d \in \mathcal{M} \setminus \mathcal{U} \subset \mathcal{M} \setminus \mathcal{K}$. Further, $d_n \in \mathcal{K}$ for each $n \in \mathbb{N}$, since every sequence in (X, d_n) is either almost stationary or d_n-converges to x_n. On the other hand, as $n \to \infty$

$$\begin{aligned} d^*(d, d_n) &= \sup_{\min\{i,j\}=n} |d(x_i, x_j) - d_n(x_i, x_j)| \\ &= \sup_{j>n} |d(x_n, x_j) - d_n(x_n, x_j)| = \sup_{j>n} \left| \frac{1}{n} - \frac{1}{j} \right| = \frac{1}{n} \to 0. \end{aligned}$$

Theorem 10 *The set $\mathcal{C}$ is nowhere dense in $(\mathcal{M}, d^*)$.*

PROOF. Suppose $\mathcal{O}_0 \cap \mathcal{C} \neq \emptyset$ and let $d \in \mathcal{O}_0 \cap \mathcal{C}$. Then by Theorem 2 $\mathcal{O}_0 \subset \mathcal{C}$ which contradicts the fact that $t_1 \in \mathcal{O}_0 \setminus \mathcal{C}$. Thus $\mathcal{O}_0 \subset \mathcal{M} \setminus \mathcal{C}$ whence $\mathcal{C} \subset \mathcal{M} \setminus \mathcal{O}_0 \subset \mathcal{M} \setminus \mathcal{H}$. Thus Lemma A implies that $\mathcal{C}$ is nowhere dense. □

Remark 4 *We do not know whether $\mathcal{C}$ is a Borel subset of $(\mathcal{M}, d^*)$.*

Let (X, K) be a linear space over the field K and define the set

$$\mathcal{N} = \mathcal{N}(X) = \{d \in \mathcal{M}(X); d(x, 0) \text{ is a norm on } X\ \}.$$

Theorem 11 *The set $\mathcal{N}$ is nowhere dense and closed in $(\mathcal{M}, d^*)$.*

PROOF. Every normed linear space is connected (cf. [3], p.148). So the nowhere density of $\mathcal{N}$ in $\mathcal{M}$ follows from Theorem 10.

Let $d \in \mathcal{M}, d_n \in \mathcal{N}$ $(n \in \mathbb{N})$ such that $d^*(d_n, d) \to 0$ $(n \to \infty)$. Then $\lim_{n \to \infty} d_n(x, 0) = d(x, 0)$ for each $x \in X$. It is now not hard to see that $d \in \mathcal{N}$. Consequently $\mathcal{N}$ is closed in $\mathcal{M}$. □

Theorem 12 *The set $\mathcal{T}$ is perfect in $(\mathcal{M}, d^*)$. Further, if $|X| \geq 3$, then $\mathcal{T}$ is nowhere dense in $\mathcal{M}$. (In case $|X| = 2$ we have $\mathcal{T} = \mathcal{M}$.)*

PROOF. It is obvious that $\mathcal{T}$ is closed in $\mathcal{M}$. Further, $\mathcal{T}$ is dense in itself since for $d \in \mathcal{T}$, $\varepsilon > 0$ $d + t_a \in K(d, \varepsilon) \cap \mathcal{T}$, where $0 < a < \varepsilon$. We shall show that $\mathcal{M} \setminus \mathcal{T}$ is dense in $\mathcal{M}$.

Let $|X| \geq 3, d \in \mathcal{T}, \varepsilon > 0$ and $\emptyset \neq A \subset X, A \neq X$. Define the metric d':

$$d'(x,y) = d'(y,x) = \begin{cases} \frac{\varepsilon}{2} & \text{if } x, y \in A, x \neq y \\ \frac{\varepsilon}{4} & \text{if } x, y \in X \setminus A, x \neq y \\ \vartheta & \text{if } x \in A, y \in X \setminus A, \frac{\varepsilon}{4} < \vartheta < \frac{\varepsilon}{2} \end{cases}$$

and for each $x \in X$ put $d'(x,x) = 0$. One can see that $d' \in \mathcal{M} \setminus \mathcal{T}$. So $d' + d \in \mathcal{M} \setminus \mathcal{T}$ and further $d' + d \in K(d, \varepsilon)$. □

If we consider the set $(0, +\infty)$ with the usual metric, then the function $F(a) = t_a$ ($a \in (0, +\infty)$) is a homeomorphism. Therefore we have:

Theorem 13 *The space $(\mathcal{T}, d^*)$ is connected.*

Let $\mathcal{A}$ and $\mathcal{B}$ denote the set of all metrics on X that are unbounded and bounded, respectively. It is proved in [4] (Theorem 5) that $\mathcal{A}$, $\mathcal{B}$ are nonempty, open subsets of the Baire space $(\mathcal{M}, d^*)$ (cf. [4], Theorem 3) provided $|X| \geq \aleph_0$. Consequently we have

Theorem 14 *The sets $\mathcal{A}, \mathcal{B}$ are sets of the 2nd category in $\mathcal{M}$ if X is infinite. (If X is finite, then $\mathcal{B} = \mathcal{M}$ and $\mathcal{A} = \emptyset$.)*

We can strengthen this theorem as follows:

Proposition 15 *The set $\mathcal{H}_a^*$ $(\mathcal{H}_a)$ is of the 2nd category in $\mathcal{M}$ for every $a > 0$.*

PROOF. Let $a > 0$. Since $\mathcal{M}$ is a Baire space, it suffices to show that $\mathcal{H}_a^*$ contains a ball. Choose an arbitrary $d_0 \in \mathcal{H}_{\frac{a}{2}}^* \subset \mathcal{H}_a^*$. Then $K(d_0, \frac{a}{2}) \subset \mathcal{H}_a^*$.

The proof for $\mathcal{H}_a$ is similar. □

Remark 5 *It is well-known that $\mathcal{K} \subset \mathcal{B}$. In this connection observe that according to Theorem 8(ii) and Theorem 14 $\mathcal{B} \setminus \mathcal{K}$ is of the 2nd category in $\mathcal{M}$, provided $|X| \geq \aleph_0$.*

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