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DIMENSIONS OF THE PERTURBED CANTOR SET

1. Introduction

Let $I_\emptyset$ =[0,1]. We obtain the left subinterval $I_{\sigma,1}$ and the right subinterval $I_{\sigma,2}$ of I_σ by deleting a middle open subinterval of I_σ inductively for each $\sigma \in \{1,2\}^n$, where $n = 0,1,2,\cdots$. Let $E_n = \cup_{\sigma\in\{1,2\}^n} I_\sigma$. Then $\{E_n\}$ is a decreasing sequence of closed sets. For each n, we set $|I_{\sigma,1}|/|I_\sigma| = a_{n+1}$ and $|I_{\sigma,2}|/|I_\sigma| = b_{n+1}$ for all $\sigma \in \{1,2\}^n$, where $|I|$ denotes the length of I. We call $F = \cap_{n=0}^\infty E_n$ a perturbed Cantor set. In particular, if $a_n = a$, $b_n = b$ for all n, we call F a Cantor type set. In this paper we assume the sequences of ratios $\{a_n\}$, $\{b_n\}$ and $\{d_n\}$, where $d_n = 1-(a_n+b_n)$, are uniformly bounded away from 0. In [1] it was shown how to find the Hausdorff dimension and the packing dimension of a Cantor type set. In this paper we investigate these dimensions for a perturbed Cantor set. We recall the s-dimensional Hausdorff measure of F, $H^s(F) = \lim_{\delta\to 0} H^s_\delta(F)$, where $H^s_\delta(F) = \inf\{\sum_{n=1}^\infty |U_n|^s : \{U_n\}_{n=1}^\infty$ is a δ-cover of $F\}$, and the Hausdorff dimension of F, $\dim_H(F) = \sup\{s > 0 : H^s(F) = \infty\} (= \inf\{s > 0 : H^s(F) = 0\})$([1]). Also we recall the s-dimensional packing measure of F, $p^s(F) = \inf\{\sum_{n=1}^\infty P^s(F_n) : \cup_{n=1}^\infty F_n = F\}$, where $P^s(F_n) = \lim_{\delta\to 0} P^s_\delta(F_n)$ and $P^s_\delta(E) = \sup\{\sum_{n=1}^\infty |U_n|^s : \{U_n\}$ is a δ-packing of $E\}$, and the packing dimension, $\dim_p(F) = \sup\{s > 0 : p^s(F) = \infty\} (= \inf\{s > 0 : p^s(F) = 0\})$([2]). We note that if $\{a_n\}$ and $\{b_n\}$ are given, then a perturbed Cantor set F is determined. We introduce functions $h^s(F) = \liminf_{n\to\infty} \prod_{k=1}^n (a_k^s + b_k^s) (= \liminf_{n\to\infty} \sum_{\sigma\in\{1,2\}^n} |I_\sigma|^s)$ and $q^s(F) = \limsup_{n\to\infty} \prod_{k=1}^n (a_k^s + b_k^s) (= \limsup_{n\to\infty} \sum_{\sigma\in\{1,2\}^n} |I_\sigma|^s)$ for $s \in (0,1)$ and F a perturbed Cantor set. Clearly $h^s(F)$ and $q^s(F)$ are decreasing functions of s. Using h^s and q^s we define the lower Cantor dimension and the upper Cantor dimension of a perturbed Cantor set F by $\dim_{\underline{C}}(F) = \sup\{s > 0 : h^s(F) = \infty\}$ and $\dim_{\overline{C}}(F) = \sup\{s > 0 : q^s(F) = \infty\}$. Then

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$\dim_{\underline{C}}(F) = \inf\{s > 0 : h^s(F) = 0\}$ and $\dim_{\overline{C}}(F) = \inf\{s > 0 : q^s(F) = 0\}$ since $h^s(F)$ and $q^s(F)$ are decreasing functions of s. We note $\dim_{\underline{C}}$ and $\dim_{\overline{C}}$ are functions whose domains are the class of the perturbed Cantor sets. Our objective is to show that the obvious covering or packing of F using the intervals of E_n as $n \to \infty$ yields the correct indices.

2. Main results

In this section F denotes a perturbed Cantor set. Before going into the investigation of the relations of $\dim_H$ and $\dim_p$ with $\dim_{\underline{C}}$ and $\dim_{\overline{C}}$, it is fruitful to know where $\dim_{\underline{C}}$ and $\dim_{\overline{C}}$ are located in (0,1) related to $\{a_n\}$ and $\{b_n\}$.

Lemma 1 $0 < \liminf_{n\to\infty} s_n \le \dim_{\underline{C}}(F) \le \dim_{\overline{C}}(F) \le \limsup_{n\to\infty} s_n < 1$, *where* $a_n^{s_n} + b_n^{s_n} = 1$.

PROOF. Since $\{a_n\}$ and $\{b_n\}$ are uniformly bounded away from 0, $0 < \liminf_n s_n \le \limsup_n s_n < 1$. To prove $\liminf_n s_n \le \dim_{\underline{C}}(F)$, we need only show that $h^s(F) = \infty$ for $0 < s < \liminf_n s_n$. Choose s and s' such that $0 < s < s' < \liminf_n s_n$. Then there exists N such that $s_k > s' > s$ for all $k \ge N$. Since there is a number $\beta < 1$ such that a_k, $b_k < \beta$ for all k,

$$\begin{aligned}\prod_{k=N}^{n}(a_k^s + b_k^s) &\ge \prod_{k=N}^{n}(a_k^{s_k} + b_k^{s_k})\beta^{s-s_k} \\ &\ge \prod_{k=N}^{n}\beta^{s-s'}.\end{aligned}$$

Then $\liminf_{n\to\infty}\prod_{k=N}^{n}(a_k^s + b_k^s) = \infty$. Hence $h^s(F) = \infty$. Similarly we have $p^t(F) = 0$ for $t > \limsup_n s_n$. □

The definitions of Cantor dimension for F show that $0 < h^s(F) < \infty$ implies that $\dim_{\underline{C}}(F) = s$ while $0 < q^s(F) < \infty$ implies that $\dim_{\overline{C}}(F) = s$. We obtain a lower bound for $\dim_H(F)$ by applying the density theorem to the natural mass distribution on F.

Theorem 2 $\dim_{\underline{C}}(F) \le \dim_H(F)$.

PROOF. Since $\{a_n\}$, $\{b_n\}$ and $\{d_n\}$ are uniformly bounded away from 0, we may assume that a_n, b_n, $d_n > \alpha$ for some $\alpha > 0$ for all n. Suppose that $0 < s < \dim_{\underline{C}}(F)$. Then $h^s(F) = \infty$. We define a set function

$$\mu(I_\sigma) = \frac{|I_\sigma|^s}{\sum_{\tau\in\{1,2\}^n}|I_\tau|^s} = \frac{|I_\sigma|^s}{\prod_{k=1}^{n}(a_k^s + b_k^s)}$$

for each $\sigma \in \{1,2\}^n$, where $n = 1, 2, \cdots$. Then μ extends to a mass distribution on [0,1] whose support is in $E = \cap_{n=1}^{\infty} E_n$ since

$$\mu(I_\sigma) = \frac{(a_{n+1}^s + b_{n+1}^s)|I_\sigma|^s}{(a_{n+1}^s + b_{n+1}^s)\sum_{\tau\in\{1,2\}^n}|I_\tau|^s} = \frac{a_{n+1}^s|I_\sigma|^s + b_{n+1}^s|I_\sigma|^s}{\sum_{\tau\in\{1,2\}^{n+1}}|I_\tau|^s}$$
$$= \mu(I_{\sigma,1}) + \mu(I_{\sigma,2})$$

(cf. Proposition 1.7 [1]). Clearly $\mu([0,1]) = 1$. Let $x \in F = \cap_{n=1}^{\infty} E_n$. Then there is a sequence $\{I_{\sigma_n}\}_{n=1}^{\infty}$, where $\sigma_n \in \{1,2\}^n$ such that $\cap_{n=1}^{\infty} I_{\sigma_n} = \{x\}$. Given a small positive number r, there exists n such that $|I_{\sigma_{n+1}}| \le r < |I_{\sigma_n}|$. Since $d_{j+1}|I_{\sigma_j}| \ge \alpha|I_{\sigma_n}| > \alpha r$ for $0 \le j \le n$, $B_{\alpha r}(x) \subset [\cup_{\tau(\neq\sigma_n)\in\{1,2\}^n} I_\tau]^c$, where $B_{\alpha r}(x)$ is the ball of radius αr with center x. Thus $\mu(B_{\alpha r}(x)) \le \mu(I_{\sigma_n})$.

For $0 < t < s$,

$$\begin{aligned}\frac{\mu(B_{\alpha r}(x))}{(\alpha r)^t} &\le \frac{\mu(I_{\sigma_n})}{\alpha^t|I_{\sigma_{n+1}}|^t}\\ &\le \frac{\mu(I_{\sigma_n})}{\alpha^t(\alpha^t|I_{\sigma_n}|^t)}\\ &= \frac{|I_{\sigma_n}|^s}{\alpha^{2t}|I_{\sigma_n}|^t\sum_{\tau\in\{1,2\}^n}|I_\tau|^s}\\ &= \frac{|I_{\sigma_n}|^{s-t}}{\alpha^{2t}\prod_{k=1}^n(a_k^s + b_k^s)}.\end{aligned}$$

Then

$$\limsup_{r\to 0}\frac{\mu(B_r(x))}{r^t} \le \limsup_{n\to\infty}\frac{|I_{\sigma_n}|^{s-t}}{\alpha^{2t}\prod_{k=1}^n(a_k^s + b_k^s)} = 0.$$

Thus $H^t(F) = \infty$ by Proposition 4.9 [1]. Hence $\dim_H(F) \ge s$. □

The similarity of $h^s(F)$ and $H^s(F)$ guarantees the following .

Theorem 3 $\dim_H(F) \le \dim_{\underline{C}}(F)$.

PROOF. Because the intervals of E_n become uniformly small as $n \to \infty$, for all $s > 0$ we have $H^s(F) \le h^s(F)$ and the result now follows from the definition. □

Corollary 4 $\dim_{\underline{C}}(F) = \dim_H(F)$.

We use a Baire Category argument to obtain a lower bound for the packing dimension of F.

Theorem 5 $\dim_{\overline{C}}(F) \le \dim_p(F)$.

PROOF. As in the proof of Theorem 2, we may assume that $a_n, b_n, d_n > \alpha$ for some $\alpha > 0$ for all n. Suppose that $0 < s < \dim_{\overline{C}}(F)$. Then $q^s(F) = \infty$. Given $\delta > 0$, for $n \geq \frac{\log \delta}{\log(1-\alpha)}$ and I_σ, where $\sigma \in \{1,2\}^n$

$$P_\delta^s(I_\sigma \cap F) \geq \sup_{k \geq n} \sum_{I_\tau \subset I_\sigma, \tau \in \{1,2\}^k} |I_\tau|^s$$
$$\geq |I_\sigma|^s \limsup_{m \to \infty} \prod_{k=n+1}^{m} (a_k^s + b_k^s) = \infty.$$

Thus $P^s(I_\sigma \cap F) = \infty$. Consider $\{F_n\}$ such that $\cup_{n=1}^\infty F_n = F$. Since F is compact, $\cup_{n=1}^\infty \overline{F_n} = F$ and there exists $\overline{F_{n_0}}$ such that the interior of $\overline{F_{n_0}}$ in F is non-empty by the Baire Category theorem. Hence there is I_σ, where $\sigma \in \{1,2\}^n$ for large n such that $I_\sigma \cap F \subset \overline{F_{n_0}}$. Then $P^s(F_{n_0}) = P^s(\overline{F_{n_0}}) \geq P^s(I_\sigma \cap F) = \infty$. Thus $P^s(F_{n_0}) = \infty$. No,

$$p^s(F) = \inf\{\sum_{n=1}^{\infty} P^s(F_n) : \cup_{n=1}^\infty F_n = F\} = \infty,$$

i.e. $\dim_p(F) \geq s$. □

Using the packing density theorem we find an upper bound for the packing dimension.

Theorem 6 $\dim_p(F) \leq \dim_{\overline{C}}(F)$.

PROOF. We may assume that a_n, $b_n > \alpha$ for some $\alpha > 0$ for all n. Suppose that $\dim_{\overline{C}}(F) < s$. Then $q^s(F) = 0$. We define a set function

$$\mu(I_\sigma) = \frac{|I_\sigma|^s}{\sum_{\tau \in \{1,2\}^n} |I_\tau|^s} = \frac{|I_\sigma|^s}{\prod_{k=1}^n (a_k^s + b_k^s)}$$

for each $\sigma \in \{1,2\}^n$, where n = 1,2,$\cdots$, as in the proof of Theorem 2. Then μ extends to a mass distribution on [0,1] whose support is in $E = \cap_{n=1}^\infty E_n$, and $\mu([0,1]) = 1$. Let $x \in F = \cap_{n=1}^\infty E_n$. Then there is a sequence $\{I_{\sigma_n}\}_{n=1}^\infty$, where $\sigma_n \in \{1,2\}^n$ such that $\cap_{n=1}^\infty I_{\sigma_n} = \{x\}$. Given a small positive number r, there exists n such that $|I_{\sigma_{n+1}}| \leq r < |I_{\sigma_n}|$. For $t > s$,

$$\frac{\mu(B_r(x))}{r^t} \geq \frac{\mu(I_{\sigma_{n+1}})}{|I_{\sigma_n}|^t}.$$

Since $|I_{\sigma_{n+1}}|/|I_{\sigma_n}| = a_{n+1}$ or $b_{n+1} > \alpha > 0$ for all n,

$$\frac{\mu(B_r(x))}{r^t} \geq \frac{\mu(I_{\sigma_{n+1}})}{(\frac{1}{\alpha})^t |I_{\sigma_{n+1}}|^t}$$
$$= \frac{\alpha^t |I_{\sigma_{n+1}}|^{s-t}}{\prod_{k=1}^{n+1}(a_k^s + b_k^s)}$$

Then

$$\liminf_{r\to 0} \frac{\mu(B_r(x))}{r^t} \geq \liminf_{n\to\infty} \frac{\alpha^t |I_{\sigma_{n+1}}|^{s-t}}{\prod_{k=1}^{n+1}(a_k^s + b_k^s)} = \infty.$$

Thus $p^t(F) = 0$ by the packing density theorem [2]. Hence $\dim_p(F) \leq s$. $\square$

Corollary 7 $\dim_{\overline{C}}(F) = \dim_p(F)$.

Applying Lemma 1 gives the following assertion.

Corollary 8 *If $a_n^{s_n} + b_n^{s_n} = 1$ defines s_n and $s_n \to s$ as $n \to \infty$, then $\dim_H(F) = \dim_p(F) = s$.*

If $\lim a_n = a$ and $\lim b_n = b$ with $0 < a$, b and $a + b < 1$, then $\{a_n\}$, $\{b_n\}$ and $\{1 - (a_n + b_n)\}$ are uniformly bounded away from 0 and $\lim s_n = s$, where $a^s + b^s = 1$. Thus we have the following Corollary.

Corollary 9 *If $\lim a_n = a$ and $\lim b_n = b$, then $\dim_H(F) = \dim_p(F) = s$, where $a^s + b^s = 1$ ($0 < a, b$ and $a + b < 1$ are assumed).*

Observation 10 *The Cantor type set F whose $a_n = a$ and $b_n = b$ for all n has Hausdorff and packing dimensions s where $a^s + b^s = 1$ ([1]).*

(Using Corollary 9 or the fact that $h^s(F) = q^s(F) = 1$, we also obtain the above result.)

Observation 11 *The Cantor-like set F whose $a_n = b_n$ for each n has Hausdorff dimension*

$$\liminf_{n\to\infty} \frac{-n\log 2}{\log \prod_{k=1}^n a_k}$$

and packing dimension

$$\limsup_{n\to\infty} \frac{-n\log 2}{\log \prod_{k=1}^n a_k}.$$

This follows from Corollaries 4 and 7 and the argument of the proof of Lemma 1 (cf.[3]).

References

[1] K. J. Falconer, Fractal geometry *(John Wiley & Sons. 1990).*

[2] S. J. Taylor and C. Tricot, *The packing measure, and its evaluation for a Brownian path.* Trans. Amer. Math. Soc. **288** (1985), 679–699.

[3] C. Tricot, *Two definitions of fractal dimension*, Math. Proc. Camb. Phil. Soc. **91** (1982), 57–74.