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CHEBYSHEV INEQUALITIES AND COMONOTONICITY

In Heinig and Maligranda [5] Chebyshev's inequality is phrased as follows. Let x and y be positive decreasing functions on $I = [0, a]$. In this case

$$\int_0^a x(s)ds \int_0^a y(s)ds \leq a \int_0^a y(s)x(s)ds$$

Letting $m(ds)$ be normalized Lebesque measure on I the above inequality may be rephrased as

$$\int_I xdm \int_I ydm \leq \int_I xydm.$$

A close inspection of their first proof reveals that m may be replaced by any probability p on I so that

$$\int_I xdp \int_I ydp \leq \int_I xydp.$$

This suggests an analogous inequality for random variables X and Y on a probability space (I, Σ, p).

$$E(X)E(Y) \leq E(XY)$$

This is well known to be equivalent to positive correlation of X and Y, that $\text{cov}(X, Y) \geq 0$. As a result, a paraphrase of Chebyshev's inequality is that two positive decreasing functions on I are positively correlated with respect to all probabilities on I. Here we extend this to give a characterization of those measurable functions on a measurable space which are positively correlated for all probabilities. This characterization is valid even for unordered measurable spaces, although order is implicit.

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If Ω is a set, two real functions f and g on Ω are said to be **comonotonic** if $f(x) > f(y)$ and $g(y) > g(x)$ is impossible for any $\{x, y\} \subset \Omega$. An equivalent condition is that $[f(x) - f(y)][g(x) - g(y)] \geq 0$ for all $\{x, y\} \subset \Omega$. A real function f induces a weak order $\succ_f$ on Ω via $x \succ_f y$ iff $f(x) > f(y)$. The associated equivalence relation is given by $x \sim_f y$ iff $f(x) = f(y)$. Two functions f and g are known to be comonotonic iff $\succ_f$ and $\succ_g$ admit a common extension to a weak order $\succ$ on Ω (That is, as subsets of $\Omega \times \Omega$, the union of the two relations $\succ_f$ and $\succ_g$ is contained in a weak order $\succ$ so that $x \succ y$ if either $f(x) > f(y)$ or $g(x) > g(y)$.) The concept of comonotonicity has become important in non-linear expected utility theory. See Armstrong [1], Wakker [7], Schmeidler [6] and Dellacherie [3].

Theorem 1 *Let (Ω, Σ) be a measurable space. Two Σ-measurable real functions f and g on Ω satisfy*

$$\int f dP \int g dP \leq \int f g dP$$

for all countably additive probabilities P iff they are comonotonic.

PROOF. One direction is the first proof given by Heinig and Maligranda of their Theorem 2.1 [5]. Suppose that f and g are comonotonic

$$\begin{aligned} 0 &\leq \int\int [f(x) - f(y)][g(x) - g(y)]P(dx)P(dy) \\ &= \int\int f(x)g(x) - f(y)g(x) - f(x)g(y) + f(y)g(y)P(dx)P(dy) \\ &= 2\int f(x)g(x)P(dx) - 2\int f(y)P(dy)\int g(x)P(dx). \end{aligned}$$

As a result,

$$\int f(x)g(x)P(dx) \geq \int f(x)P(dx)\int g(x)P(dx).$$

For the other direction suppose f and g are positively correlated with respect to P then

$$\frac{1}{2}[f(x)g(x) + f(y)g(y)] \geq \frac{1}{4}[f(x) + f(y)][g(x) + g(y)].$$

Simple algebra now yields

$$f(x)g(x) - f(x)g(y) - f(y)g(x) + f(y)g(y) \leq 0 \text{ or}$$
$$[f(x) - f(y)][g(x) - g(y)] \geq 0.$$

□

Remark 1 *Theorem 2.1 of Heinig-Maligranda [5] dates back to at least Chebyshev [2].*

Recall that a Radon measure on a compact Hausdorff space K is an element of $M(K) = C'(K)$, or alternatively, is an inner regular Borel measure or a countably additive Baire measure.

Corollary 1 *Let K be a compact Hausdorff space. Continuous functions f and g are positively correlated with respect to all probability Radon measures iff they are comonotonic.*

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