Real Analysis Exchange
Vol. 19(1), 1993/94, pp. 254–255

Jack W. Lamoreaux and Gerald M. Armstrong, Department of Mathematics, Brigham Young University, Provo, Utah 84602

A NECESSARY AND SUFFICIENT CONDITION FOR GAUGE INTEGRABILITY

Throughout, the words "integrable" and "integral" mean "gauge integrable" and "gauge integral," respectively.

Theorem 1 *A function f on $[a,b]$ is integrable in $[a,b]$ if and only if the following condition is satisfied: there is a function F on $[a,b]$ and a strictly increasing differentiable function ϕ mapping $[\alpha,\beta]$ onto $[a,b]$ such that $(F \circ \phi)' = (f \circ \phi) \cdot \phi'$ in $[\alpha,\beta]$. In this case, F is the indefinite integral of f in $[a,b]$.*

PROOF. If the condition of the theorem is satisfied, let $\lambda(x) = x$ for every real number x, and select a $t \in [\alpha,\beta]$. According to the Fundamental Theorem of Calculus ([2, p. 43]), $(f \circ \phi) \cdot \phi'$ is integrable in $[\alpha,\beta]$ with respect to λ and

$$F(\phi(t)) - F(\phi(\alpha)) = \int_\alpha^t (f \circ \phi) \cdot \phi' \, d\lambda.$$

Since ϕ is the indefinite integral of ϕ', it follows from the proposition on p. 186 of [2] (whose proof can be found in §2 on p. 264) that $f \circ \phi$ is integrable in $[\alpha,\beta]$ with respect to ϕ and

$$\int_\alpha^t f \circ \phi \, d\phi = \int_\alpha^t (f \circ \phi) \cdot \phi' \, d\lambda.$$

By the proposition on p. 207 of [2], the function f is integrable in $[a,b]$ with respect to λ and

$$\int_{\phi(\alpha)}^{\phi(t)} f d\lambda = \int_\alpha^t f \circ \phi \, d(\lambda \circ \phi) = \int_\alpha^t f \circ \phi \, d\phi.$$

Key Words: guage integral
Mathematical Reviews subject classification: Primary: 26A42
Received by the editors June 6, 1992

Now letting $x = \phi(t)$, we obtain $F(x) - F(a) = \int_\alpha^x f d\lambda$, which means that F is the indefinite integral of f in $[a, b]$.

Conversely, if f is integrable in $[a, b]$ and F is the indefinite integral of f, then F is ACG_* by [1]. Using [4], we can find a strictly increasing differentiable function g mapping an interval $[\mu, \nu]$ onto $[a, b]$ and such that $F \circ g$ is differentiable in $[\mu, \nu]$. There is a negligible set $E \subset [a, b]$ with $F'(x) = f(x)$ for each $x \in [a, b] \setminus E$. According to [5], we can find a strictly increasing function h mapping an interval $[\alpha, \beta]$ onto $[\mu, \nu]$ and such that $h'(t) = 0$ whenever t belongs to the set $T = h^{-1}(E)$. Set $\phi = g \circ h$, and observe that on $[\alpha, \beta]$ we have

$$\begin{aligned} (F \circ \phi)' &= [(F \circ g)' \circ h] \cdot h' \\ (f \circ \phi) \cdot \phi' &= (f \circ g \circ h) \cdot (g' \circ h) \cdot h' = ([(f \circ g) \cdot g'] \circ h) \cdot h'. \end{aligned}$$

In particular, $(F \circ \phi)' = (f \circ \phi) \cdot \phi' = 0$ on T. Since $(F \circ g)' = (f \circ g) \cdot g'$ on $[\alpha, \beta] \setminus T$, we see that the equality $(F \circ \phi)' = (f \circ \phi) \cdot \phi'$ holds everywhere in $[\alpha, \beta]$.

We remark that the derivative condition in the theorem was studied by G. P. Tolstov [3]. If f and F are functions defined on $[a, b]$, and there exists an increasing differentiable function ϕ from $[\alpha, \beta]$ onto $[a, b]$ such that $(F \circ \phi)' = (f \circ \phi) \cdot \phi'$ on $[\alpha, \beta]$, Tolstov called f a *parametric derivative* of F on $[a, b]$. We note that f is unique a.e., and also it is not hard to see that ϕ can be taken to be strictly increasing on $[\alpha, \beta]$.

Thus our theorem can be restated as: f is integrable, with F as its indefinite integral, if and only if f is a parametric derivative of F.

References

[1] R. Gordon, *Equivalence of the Generalized Riemann and Restricted Denjoy Integrals*, Real Analysis Exchange **12** (1986–87), 551–574.

[2] R. M. McLeod, *The Generalized Riemann Integral*, Mathematical Association of America (1980), Providence, R.I.

[3] G. P. Tolstov, *Parametric Differentiation and the Restricted Denjoy Integral*, Mat. Sb. **53** (1961), 387–392.

[4] G. P. Tolstov, *La methode de Perron pour l'integrale de Denjoy*, Mat. Sb. N.S. **8, 50** (1940), 149–168

[5] Zahorski, *Sur la Première Dérivée*, Trans. Amer. Math. Soc. **69** (1950), 1–54.