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# SUFFICIENT CONDITIONS FOR ANALYTICITY

## 1. Introduction

A real valued function $f$, infinitely differentiable on an interval $I$, is said to be *analytic at $t$ in $I$*, if the Taylor series for $f$ about $t$ converges to $f(x)$ for $x$ in some neighborhood of $t$ and $f$ is *analytic on $I$* if it is analytic at every point of $I$. Thus a function analytic on a closed interval $[a, b]$ can be extended to be analytic on an open interval containing $[a, b]$.

Sufficient conditions for analyticity exist in abundance, particularly those which arise from the theory of *absolutely monotonic functions*, i.e. functions which, along with all their derivatives, are non-negative. A survey of these results is available in [4]. The purpose of this paper is to present other sufficient conditions which also arise as extensions of absolutely monotonic functions and extend the results of [1], [2], and [3]. Specifically, there are two parts to this paper. In the first we show that $[\phi^n(x)f(x)]^{(n)} \geq 0$ for all $n \geq 0$ on $[a, b]$ is a sufficient condition for analyticity on $[a, b)$ where $\phi$ is an entire function of a general class. In the second part, a sequence of differential operators introduced in [7] leading to a generalization of Taylor series will also provide a sufficient condition for analyticity when they are non-negative on the same set.

A well-known theorem of much use in the following, normally attributed to Pringsheim [6], states that $f$ is analytic on $[a, b]$ if and only if there are constants $M$, $r$ with $|f^{(n)}(x)| \leq M\ r^n n!$ on $[a, b]$, for all $n \geq 0$. Actually Pringsheim's proof was faulty. A correct proof can be found in [5]. Nevertheless, for easy reference, this result will still be referred to as Pringsheim's theorem.

Bernstein [1] proved that a function is analytic on $[a, b)$ if it is absolutely monotonic on $[a, b]$. In fact $f$ can be extended to be analytic on the set of

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$x$ for which $|x - a| < b - a$ and moreover its Taylor expansion about $x = a$ converges with a radius at least $b - a$. Thus a function absolutely monotonic on $[0, \infty)$ can be extended to be analytic on $(-\infty, \infty)$ and representable by its Taylor series about $x = 0$ convergent on $(-\infty, \infty)$. It is thus the restriction of an entire function to the non-negative real axis.

An obvious comment is that a power series (or polynomial) in powers of $x$ is absolutely monotonic on $[0, \infty)$ if and only if all of its coefficients are non-negative.

## 2. Extensions I

The requirement that $f$ be absolutely monotonic can be relaxed. Recently, [2], it was shown that

$$[e^{knx} f(x)]^{(n)} \geq 0 \tag{2.1}$$

on $[a, b]$ for all $n \geq 0$ is a sufficient condition for the analyticity of $f$ on $[a, b)$ where $k$ is any positive constant. This suggests that it may be possible to "boost" a $C^\infty$ function $f$ by an auxiliary sequence of $C^\infty$ functions $\{\phi_n\}$ and use the condition $[\phi_n(x) f(x)]^{(n)} \geq 0$ to obtain analyticity. It turns out that a good choice occurs when $\phi_n(x)$ has the form $\phi_n(x) = \phi^n(x)$ for a special class of functions $\phi$ which are now introduced.

**Definition 1** *An entire function $\phi$ is said to be in class $\mathbb{A}$ if it is of the form:*

$$\phi(z) = e^{g(z)} \prod_1^\infty \left(1 + \frac{z}{\alpha_n}\right), \quad \sum_1^\infty \frac{1}{|\alpha_n|} < \infty \tag{2.2}$$

*where either $\alpha_n > 0$ or the $\alpha_n$ occur in complex conjugate pairs with $Re(\alpha_n) \geq 0$. Also $g$ is an entire function with $g(0) = 0$ and, when restricted to the real axis, is absolutely monotonic on $[0, \infty)$.*

Finite products in (2.2) are possible by having $\alpha_n = \infty$ for some $n$. Note that $\phi(z) \equiv 1$ is included. From the comments above we know that the coefficients in the Taylor series of $g$ about $z = 0$ are non-negative.

We observe that $\phi$ is real for real $z$. Also since the factors coming from conjugate pairs of roots contribute

$$\left(1 + \frac{z}{\alpha_n}\right)\left(1 + \frac{z}{\bar{\alpha}_n}\right) = 1 + \frac{2z[Re(\alpha_n)] + z^2}{|\alpha_n|^2}$$

and thus give rise to non-negative coefficients of powers of $z$, we see that the resulting Taylor expansion for $\phi$ about $z = 0$ contains only non-negative coefficients because of this and the other factors in (2.2). Thus $\phi$, when restricted to the real line, is absolutely monotonic on $[0, \infty)$.

Three preliminary lemmas are needed.

**Lemma 2.1** *For $\phi$ in class $\mathbb{A}$ with $g(z) \equiv 0$, let $P_m(x) = \prod_{i=1}^m \left(1 + \frac{x}{\alpha_i}\right)$. Then for each fixed $n \geq 1$ and $k \geq 0$ the sequence (on $m$) $\{[P_m^{-n}(x)]^{(k)}\}$ converges to $[\phi^{-n}(x)]^{(k)}$ for all $x \geq 0$.*

This rather technical result is a consequence of the elementary theory of analytic functions of one complex variable. The sequence $\{P_m(x)\}$ converges uniformly to $\phi(z)$ on compact sets in the plane and thus the sequence (on $m$) $\{P_m^n(z)\}$ converges uniformly to $\phi^n(z)$ on the same sets for each $n \geq 1$. Restricting ourselves now to sets of the form $S_\alpha = \{z : |\arg(z)| \leq \alpha\}$, for $\alpha < \pi/2$, (these are closed angular sectors containing the origin and $\phi(z) \neq 0$ there), then $P_m(z)$ does not vanish in these sets so that $\{P_m^{-1}(z)\}$ will converge uniformly on compact subsets of $S_\alpha$ to $[\phi(z)]^{-1}$ and hence the same applies to $\{P_m^{-n}(z)\}$ converging uniformly to $\phi^{-n}(z)$ for each $n \geq 1$. From the same general theory, we know that derivatives also converge so that $\{[P_m^{-n}(z)]^{(k)}\}$ converges to $[\phi^{-n}(z)]^{(k)}$ as $m \to \infty$ for all $z$ in $S_\alpha$ and in particular for $z = x \geq 0$.

**Lemma 2.2** *Let $\phi$ be in class $\mathbb{A}$. Then for every $R > 0$, there is a constant $M$ for which for $0 \leq x \leq R$, $n = 1, 2, \ldots$, and $0 \leq k \leq n$,*

$$|(\phi^{-n}(x))^{(k)}| \leq M^k n^k.$$

This will be proved in stages. First assume that $g(z) \equiv 0$ in (2.2) and $1 \leq k \leq n$. Then for $x \geq 0$,

$$\left| \left[ \left(1 + \frac{x}{\alpha_1}\right)^{-n} \right]^{(k)} \right| = \frac{n(n+1)\cdots(n+k-1)}{\left|\left(1 + \frac{x}{\alpha_1}\right)\right|^{n+k} |\alpha_1|^k} \leq \frac{(2n)^k}{|\alpha_1|^k}$$

because

$$\left|1 + \frac{x}{\alpha_1}\right| \geq \left|Re\left(1 + \frac{x}{\alpha_1}\right)\right| = \left(1 + \frac{xRe(\alpha_1)}{|\alpha_1|^2}\right) \geq 1$$

using the definition of class $\mathbb{A}$. By Leibniz' Rule

$$\left| \left[ \left(1 + \frac{x}{\alpha_1}\right)^{-n} \left(1 + \frac{x}{\alpha_2}\right)^{-n} \right]^{(k)} \right|$$

$$= \left| \sum_{i=0}^{k} \binom{k}{i} \left[ \left(1 + \frac{x}{\alpha_i}\right)^{-n} \right]^{(i)} \left[ \left(1 + \frac{x}{\alpha_2}\right)^{-n} \right]^{(k-i)} \right|$$

$$\leq \sum_{i=0}^{k} \binom{k}{i} \frac{(2n)^i}{|\alpha_1|^i} \frac{(2n)^{k-i}}{|\alpha_2|^{k-i}} = (2n)^k \left(\frac{1}{|\alpha_1|} + \frac{1}{|\alpha_2|}\right)^k.$$

It follows by induction that

$$\left|\left[\left(\prod_{i=1}^{m}\left(1+\frac{x}{\alpha_i}\right)\right)^{-n}\right]^{(k)}\right| \le (2n)^k\left(\sum_{i=1}^{m}\frac{1}{|\alpha_i|}\right)^k \le (2An)^k$$

where $A = \sum_{n=1}^{\infty}\frac{1}{|\alpha_n|}$ is finite. The expression inside the absolute value converges to $[\phi^{-n}(x)]^{(k)}$ as $m \to \infty$ by Lemma 2.1. Thus

$$|(\phi^{-n}(x))^{(k)}| \le (2An)^k$$

valid for all $x \ge 0$, all positive integers $n$, and $1 \le k \le n$. It also clearly holds for $k = 0$ since $\phi(x) \ge 1$.

We now assume that $\phi(x) = e^{g(x)}$ and show the same result however with $x$ restricted. Since $g$ is entire, then for every $R > 0$, there are constants $B$, $r$ with $|g^{(k)}(x)| \le Br^k k!$ for $0 \le x \le R$ and $k = 0, 1, 2, \dots$ . This a consequence of Pringsheim's theorem. We may, in fact, choose $B$, $r$ so large that $|g^{(k)}(x)| \le Br^{k-1}(k-1)!$ for $k = 1, 2, \dots$ and $0 \le x \le R$. It will be more convenient to use this form in what follows.

Now let $h(x) = \phi^{-n}(x) = e^{-ng(x)}$. Then $h'(x) = -ng'(x)e^{-ng(x)}$ and by Leibniz' rule,

$$h^{(k+1)}(x) = -n\sum_{i=0}^{k}\binom{k}{i}h^{(k-i)}(x)g^{(i+1)}(x).$$

Choose any $M > B+r$. We will show by induction on $k$ that $|h^{(k)}(x)| \le M^k n^k$ whenever $0 \le x \le R$ and $0 \le k \le n$. The inequality obviously holds when $k = 0$ for any $x$ and $n$. Then

$$|h^{(k+1)}(x)| \le n\sum_{i=0}^{k}\binom{k}{i}M^{k-i}n^{k-i}Br^i i!$$

and then

$$|h^{(k+1)}(x)| \le M^{k+1}n^{k+1}B\sum_{i=0}^{k}\frac{k!}{(k-i)!}M^{-i-1}n^{-i}r^i.$$

Now use $\frac{k!}{(k-i)!} \le k^i \le n^i$ to get

$$\begin{aligned}|h^{(k+1)}(x)| &\le M^{k+1}n^{k+1}\frac{B}{M}\sum_{i=0}^{k}\left(\frac{r}{M}\right)^i \le M^{k+1}n^{k+1}\frac{B}{M}\sum_{i=0}^{\infty}\left(\frac{r}{M}\right)^i \\ &= M^{k+1}n^{k+1}\frac{B}{M-r}.\end{aligned}$$

Here we have used the formula for the sum of a geometric series. Since $M-r > B$, the induction is complete.

For the general case, write $\phi(x) = e^{g(x)}\theta(x)$ where $\theta(x)$ is of the form considered in the first part of this proof. Thus $\phi^{-n}(x) = h(x)\theta^{-n}(x)$ with $h$ as in the previous paragraph. By the work just completed, for each $R > 0$, there are constants $M_1, M_2$ for which $|h^{(k)}(x)| \leq {M_1}^k n^k$ and $|[\theta^{-n}(x)]^{(k)}| \leq {M_2}^k n^k$ for $0 \leq x \leq R$, $k = 0, 1, 2, \ldots$ and $n = 1, 2, \ldots$ . Then Leibniz' Rule leads to

$$\begin{aligned} \left|(\phi^n(x))^{(k)}\right| &= \left|\sum_{i=0}^{k}\binom{k}{i}(\theta^{-n}(x))^{(i)}h^{(k-i)}(x)\right| \\ &\leq \sum_{i=0}^{k}\binom{k}{i}(M_2 n)^i(M_1 n)^{k-i} = [(M_2+M_1)n]^k \end{aligned}$$

and this proves the lemma.

**Lemma 2.3** *Let $\phi$ be in class $\mathbb{A}$ and $[\phi^n(x)f(x)]^{(n)} \geq 0$ on $[a,b]$ for $n = 0, 1, \ldots$ with $a \geq 0$. Then $[\phi^n(x)f(x)]^{(k)} \geq 0$ on $[a,b]$ for $n = 0, 1, \ldots$ and $0 \leq k \leq n$.*

This will follow by induction on $n$. The result is clear for $n = 0$ and assume it true for some $n$ and all $k$, $0 \leq k \leq n$. Then

$$[\phi^{n+1}(x)f(x)]^{(k)} = [\phi(x)\phi^n(x)f(x)]^{(k)} = \sum_{i=0}^{k}\binom{k}{i}\phi^{(k-i)}(x)[\phi^n(x)f(x)]^{(i)} \geq 0$$

using the induction hypothesis for $k = 0, 1, \ldots, n$ and the fact that $\phi$ is absolutely monotonic on $[0, \infty)$. The case $k = n+1$ is a consequence of the hypothesis. The proof is complete.

We can now obtain the main result of the section.

**Theorem 2.4** *Let $f$ be infinitely differentiable on $[a,b]$. If $[\phi^n(x-a)f(x)]^{(n)} \geq 0$ on $[a,b]$ for $n = 0, 1, \ldots$, for some function $\phi$ in class $\mathbb{A}$, then $f$ is analytic on $[a,b)$.*

The use of $\phi(x-a)$ is due to the fact that the desirable properties of $\phi$ occur on the non-negative reals. It is clear that the condition specified can be replaced by $[\phi^n(x)f(x+a)]^{(n)} \geq 0$ on $[0, b-a]$ and $f$ will be analytic on $[a,b)$ if $g$, defined by $g(x) = f(x+a)$, is analytic on $[0, b-a)$. Thus it will be no restriction to assume in the proof that the interval $[a,b]$ is contained in the non-negative reals and $[\phi^n(x)f(x)]^{(n)} \geq 0$ on $[a,b]$ for all $n \geq 0$.

To prove the theorem, we use Taylor's Theorem. For $x, t$ in $[a, b]$

$$\phi^{n+1}(x)f(x)$$
$$= \sum_{k=0}^{n} [\phi^{n+1}(t)f(t)]^{(k)} \frac{(x-t)^k}{k!} + \frac{1}{n!} \int_t^x [\phi^{n+1}(u)f(u)]^{(n+1)}(x-u)^n du.$$

By hypothesis and Lemma 2.3, every term on the right side is non-negative for $a \le t < x \le b$. Thus for these values

$$0 \le [\phi^{n+1}(t)f(t)]^{(k)} \frac{(x-t)^k}{k!} \le \phi^{n+1}(x)f(x) \le AB^{n+1}$$

where $A, B$ are bounds for $f$ and $\phi$ on $[a, b]$ respectively. Hence

$$0 \le [\phi^{n+1}(t)f(t)]^{(k)} \le AB^{n+1}(x-t)^{-k}k! \tag{2.3}$$

for $a \le t < x \le b$ and $k = 0, 1, \ldots, n$. Then we obtain

$$\begin{aligned} |f^{(n)}(t)| &= \left| \left[ \phi^{n+1}(t)f(t) \cdot \frac{1}{\phi^{n+1}(t)} \right]^{(n)} \right| \\ &= \left| \sum_{k=0}^{n} \binom{n}{k} \left(\phi^{-n-1}(t)\right)^{(k)} \left[\phi^{n+1}(t)f(t)\right]^{(n-k)} \right| \end{aligned}$$

and by (2.3) and Lemma 2.2 this is (we choose $R$ so that $0 \le a \le x \le b \le R$)

$$\begin{aligned} &\le \sum_{k=0}^{n} \binom{n}{k} M^k (n+1)^k AB^{n+1}(x-t)^{-n+k}(n-k)! \\ &\le AB^{n+1} n! (x-t)^{-n} \sum_{k=0}^{\infty} \frac{[M(n+1)(x-t)]^k}{k!} \\ &= AB^{n+1} n! (x-t)^{-n} e^{M(n+1)(x-t)} = An!(x-t) \left[ \frac{Be^{M(x-t)}}{x-t} \right]^{n+1} \end{aligned}$$

Thus for $a \le t \le c < x = b$,

$$|f^{(n)}(t)| \le An!(b-a) \left[ \frac{Be^{M(b-a)}}{b-c} \right]^{n+1} = Kr^n n!$$

for some constants $K, r$, from which we conclude by Pringsheim's theorem that $f$ is analytic on $[a, c]$. Since $c$ is arbitrary, $c < b$, then the result follows for $[a, b)$.

It also follows from this proof that if the hypothesis holds on $[a,\infty)$ for some $a \geq 0$, then $f$ is analytic on $[a,\infty)$.

Besides $\phi(x) \equiv 1$ (giving Bernstein's result), the most interesting cases would seem to occur when $\phi(x) = e^{kx}$ for some positive constant $k$, and when $\phi(x)$ is a polynomial. The former was proved in [2] where it was also shown that it is best possible in the sense that $[e^{knx}f(x)]^{(n)} \geq 0$ can not be replaced by $[e^{kn^\alpha x}f(x)]^{(n)} \geq 0$ for any $\alpha > 1$.

## 3. Extensions II.

In this section, some early work of Widder [7] will be the basic source. For $f$ infinitely differentiable on $[a,b]$, define an operator $L_n$ by $L_0[f(x)] = f(x)$ and for $n \geq 1$,

$$\begin{aligned} L_n[f(x)] &= \phi^{-n}(x)D[\phi^n(x)L_{n-1}[f(x)]] \\ &= \phi^{-n}(x)D[\phi(x)D[\phi(x)\ldots D[\phi(x)f(x)]\ldots]]. \end{aligned} \tag{3.1}$$

We again assume without loss of generality that $a \geq 0$ while $\phi$ is a function of class $\mathbb{A}$ defined above. A companion set of functions is defined by

$$g_n(x,t) = \phi^{n+1}(t)\int_t^x \phi^{-n-1}(u)g_{n-1}(x,u)du,\ n \geq 1,\ g_0(x,t) = \frac{\phi(t)}{\phi(x)}. \tag{3.2}$$

Let

$$\begin{aligned} R_n(x,t) &= \int_t^x g_n(x,u)L_{n+1}[f(u)]du \\ &= \int_t^x \phi^{-n-1}(u)g_n(x,u)\phi^{n+1}(u)L_{n+1}[f(u)]du. \end{aligned} \tag{3.3}$$

Then an integration by parts shows that for $n \geq 1$,

$$R_n(x,t) = -L_n[f(t)]g_n(x,t) + R_{n-1}(x,t).$$

Repeating a process which generalizes the usual proof of Taylor's Theorem, we get

$$\begin{aligned} R_n(x,t) &= -L_n[f(t)]g_n(x,t) - L_{n-1}[f(t)]g_{n-1}(x,t) \\ &\quad - \cdots - L_1[f(t)]g_1(x,t) + R_0(x,t). \end{aligned}$$

Since $R_0(x,t) = f(x) - g_0(x,t)L_0[f(t)]$, then (cf. [7;136])

$$f(x) = \sum_{j=1}^{n} L_j[f(t)]g_j(x,t) + R_n(x,t). \tag{3.4}$$

This suggests the possibility of expanding $f(x)$ in a series of the form, $\sum_{k=0}^{\infty} L_k[(t)]g_k(x,t)$, an approach used in [7] but not here. Note that if $\phi(x) \equiv 1$ then $L_n[f(x)] = D^n f(x)$ and $g_n(x,t) = \frac{(x-t)^n}{n!}$ so that (3.4) becomes Taylor's theorem.

Some preliminary results are needed. Observe first that

$$g_1(x,t) = \frac{\phi^2(t)}{\phi(x)} \int_t^x \frac{1}{\phi(u)} du, \quad g_2(x,t) = \frac{\phi^3(t)}{\phi(x)} \int_t^x \frac{1}{\phi(u)} du \int_u^x \frac{1}{\phi(v)} dv$$

and a change in the order of integration leads to

$$g_2(x,t) = \frac{\phi^3(t)}{\phi(x)} \int_t^x \frac{1}{\phi(v)} dv \int_t^v \frac{1}{\phi(u)} du.$$

We deduce easily that, for $k \geq 1$,

$$g_k(x,t) = \frac{\phi^{k+1}(t)}{\phi(x)} \int_t^x \frac{1}{\phi(u_k)} du_k \cdots \int_t^{u_2} \frac{1}{\phi(u_1)} du_1. \tag{3.5}$$

Although these functions have two variables, it is convenient to use the notations: $g_k^{(m)}(x,t) = \frac{\partial^m g_k(x,t)}{\partial x^m}$ and $g_k^{(m)}(x,x) = g_k^{(m)}(x,t) \mid_{t=x}$. This notation will also be used for other multivariable functions.

It is not difficult to see from (3.5) that

$$g_k^{(m)}(x,x) = 0, \quad 0 \leq m < k, \quad g_k^{(k)}(x,x) = 1. \tag{3.6}$$

**Lemma 3.1** *If $f$ is infinitely differentiable on $[a,b]$, then for all $n \geq 0$,*

$$\begin{aligned} f^{(n)}(x) &= L_0[f(x)]g_0^{(n)}(x,x) + L_1[f(x)]g_1^{(n)}(x,x) \\ &\quad + \cdots + L_{n-1}[f(x)]g_{n-1}^{(n)}(x,x) + L_n[f(x)]. \end{aligned}$$

This is proved in [7;131] in a more general setting but follows quickly from (3.4), since it suffices to show that $R_n^{(n)}(x,x) = 0$. Repeated differentiations of the first equation in (3.3) and use of (3.6) results in

$$R_n^{(n-1)}(x,t) = \int_t^x g_n^{(n-1)}(x,u) L_{n+1}[f(u)] du.$$

Another differentiation gives us $R_n^{(n)}(x,x) = g_n^{(n-1)}(x,x)L_{n+1}[f(x)] = 0$ by (3.6).

We will need bounds for the factors that occur in the formula.

**Lemma 3.2** *For $\phi$ in class $\mathbb{A}$, there are constants $A, C$ so that for $0 \le a \le x \le b$ and $0 \le k \le n$,*

$$|g_k^{(n)}(x,x)| \le AC^n(k+1)^{n-k}. \tag{3.7}$$

We note from (3.5) that $g_k(x,t)$ is the product of $\frac{\phi(t)}{\phi(x)}$ and a function whose derivative with respect to $x$ is $g_{k-1}(x,t)$. Thus we may apply Leibniz' rule for differentiating a product and use (3.6) to get, for $k \ge 1$,

$$g_k^{(n)}(x,x) = \phi(x) \sum_{j=k}^{n} \binom{n}{j} \left(\frac{1}{\phi(x)}\right)^{(n-j)} g_{k-1}^{(j-1)}(x,x). \tag{3.8}$$

On $[a,b]$, $|\phi(x)| \le A$, for some $A$, and we can assume that $A > 1$. For the case $n = k = 0$, Equation (3.2) gives us $g_0(x,x) = 1$. We will also use $\left|\left(\frac{1}{\phi(x)}\right)^{(j)}\right| \le B^j$ for some constant $B$, a consequence of Lemma 2.2. We now proceed by induction. Choose $C$ larger than $B$ and $Ae$ and assume that (3.7) holds for $g_k^{(m)}(x,t)$ for all $m < n$ and $0 \le k \le m$. Then from (3.8) for $1 \le k \le n$ ($k = 0$ is a special case),

$$\begin{aligned}
|g_k^{(n)}(x,x)| &\le A\sum_{j=k}^{n}\binom{n}{j}B^{n-j}AC^{j-1}k^{j-k} \le A^2\sum_{j=k}^{n}\binom{n}{j}C^{n-j}C^{j-1}k^{j-k} \\
&= A^2C^{n-1}k^{-k}\sum_{j=k}^{n}\binom{n}{j}k^j \le A^2C^{n-1}k^{-k}\sum_{j=0}^{n}\binom{n}{j}k^j \\
&= A^2C^{n-1}k^{-k}(k+1)^n \le AC^n(k+1)^{n-k}.
\end{aligned}$$

The validity of the last step is a consequence of $A\left(1+\frac{1}{k}\right)^k \le Ae \le C$. The exceptional case, $k = 0$, follows from (3.2)

$$|g_0^{(n)}(x,x)| = \phi(x)\left|\left(\frac{1}{\phi(x)}\right)^{(n)}\right| \le AB^n \le AC^n.$$

This completes the proof.

**Lemma 3.3** *If $\phi$ is in class $\mathbb{A}$, then for $0 \le t < x$ and all $n \ge 0$,*

$$g_n(x,t) \ge \left(\frac{\phi(t)}{\phi(x)}\right)^{n+1} \frac{(x-t)^n}{n!}.$$

A consequence of the definition of class $\mathbb{A}$ and absolute monotonicity is that the function $\phi$ is positive and non-decreasing on $[0,\infty)$. The case $n = 0$

follows from the definition of $g_0(x,t)$ in (3.2). Now assume the result true for $n$ replaced by $n-1$. Then from (3.2),

$$\begin{aligned} g_n(x,t) &\geq \phi^{n+1}(t) \int_t^x \phi^{-n-1}(u) \frac{\phi^n(u)}{\phi^n(x)} \frac{(x-u)^{n-1}}{(n-1)!} du \\ &\geq \left(\frac{\phi(t)}{\phi(x)}\right)^{n+1} \int_t^x \frac{(x-u)^{n-1}}{(n-1)!} du \end{aligned}$$

and an integration completes the proof.

**Theorem 3.4** *Let $f$ be infinitely differentiable on $[a,b]$. If there is a function $\phi$ in class $\mathbb{A}$ for which $L_n[f(x+a)] \geq 0$ on $[0, b-a]$ for $n = 0, 1, \ldots,$ then $f$ is analytic on $[a,b)$.*

As mentioned earlier, we can assume $a \geq 0$ and $L_n[f(x)] \geq 0$ on $[a,b]$. Since every term in the expression in Equation (3.4) is non-negative when $0 \leq a \leq t < x \leq b$, we obtain $L_n[f(t)] g_n(x,t) \leq f(x) \leq M$ for some constant $M$. An appeal to Lemma 3.3 gives us

$$0 \leq L_n[f(t)] \leq M \left(\frac{\phi(x)}{\phi(t)}\right)^{n+1} n!(x-t)^{-n}.$$

Restricting ourselves to a $\leq t \leq c < b$ and letting $x = b$, we get

$$0 \leq L_n[f(t)] \leq MK^{n+1} n!(b-c)^{-n}, \quad K = \frac{\phi(b)}{\phi(a)}.$$

Now it follows from the result in Lemma 3.1 that

$$|f^{(n)}(x)| \leq \sum_{j=0}^{n} L_j[f(x)] |g_j^{(n)}(x,x)| \leq \sum_{j=0}^{n} MK^{j+1} j!(b-c)^{-j} AC^n (j+1)^{n-j}.$$

We have also used Lemma 3.2 and Equation (3.6). This inequality is simplified by using $(j+1)^{n-j} \leq (j+1)(j+2)\cdots(n) = \frac{n!}{j!}$ to obtain

$$|f^{(n)}(x)| \leq AMKC^n n! \sum_{j=0}^{n} \left(\frac{K}{b-c}\right)^j \leq r^n n!$$

valid for some constant $r \geq C$ because the sum is a geometric sum. Now we apply Pringsheim's theorem to conclude that $f$ is analytic on $[a,c]$ for arbitrary $c < b$. The proof is now complete.

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