Real Analysis Exchange
Vol. 19(1), 1993/94, pp. 98–105

Kenneth Schilling, Department of Mathematics, University of Michigan-Flint, Flint, MI 48503

A CATEGORY BASE FOR MYCIELSKI'S IDEALS

Given sets $S \subset 2^\omega$ and $K \subset \omega$, the infinite game of perfect information $\Gamma(S, K)$ is played as follows: Players I and II choose consecutive terms of a sequence $x = (x_0, x_1, x_2, \cdots) \in 2^\omega$, player I choosing x_i for $i \in K^c$, player II choosing x_i for $i \in K$. Player I wins if $x \in S$, player II otherwise.

Now let $M = (K_s : s \in Sq)$ be a system of subsets of ω such that $K_{s0} \cup K_{s1} \subset K_s$ and $K_{s0} \cap K_{s1} = \phi$. ($Sq$ is the set of finite sequences of 0's and 1's). We further assume that each K_s is infinite and has infinite complement. In [1], J. Mycielski defined the translation-invariant $\sigma-$ideal I_M on 2^ω by putting $S \in I_M$ if, for all $s \in Sq$, player II has a winning strategy for the game $\Gamma(S, K_s)$.

Our purpose here is to try to solve the equation

$$\frac{\text{BAIRE PROPERTY}}{\text{FIRST CATEGORY}} = \frac{?}{I_M}.$$

In other words, if we think of the I_M-sets as being, in some sense, of the first category, then which sets have the Baire property in this sense? More precisely, we want to find a σ-algebra $\mathcal{B}$ on 2^ω which contains the Borel sets and includes I_M, such that the quotient algebra $\mathcal{B}/I_M$ is a complete Boolean algebra and a regular subalgebra of the Boolean algebra $\mathcal{P}(2^\omega)/I_M$. (This is an analogue of a well-known theorem of Birkhoff and Ulam of general topology. See section 1.1.)

As we shall see, no such $\mathcal{B}$ exists. We therefore modify the problem. Do there exist a σ-ideal $\mathcal{M}$ and a σ-algebra $\mathcal{B}$ on 2^ω such that $\mathcal{M}$ contains the same Borel sets as I_M, $\mathcal{B}$ contains the Borel sets and includes $\mathcal{M}$, and $\mathcal{B}/\mathcal{M}$ is a complete Boolean algebra and a regular subalgebra of $\mathcal{P}(2^\omega)/\mathcal{M}$?

Assuming the continuum hypothesis, there does. (We don't know the answer in ZFC alone.) Our construction uses J. Morgan's theory of category bases. (We give a brief introduction to the theory of category bases in the

Mathematical Reviews subject classification: 1991 54A05
Received by the editors June 1, 1992

next section. For a real introduction, see [2] or [3].) Category bases generalize the topological theory of category. We shall introduce a category base $\mathcal{C}^*_M$ on 2^ω. Our $\mathcal{B}$ and $\mathcal{M}$ will be, respectively, the Baire property and meager sets with respect to this new category base.

1. Preliminaries

We write Sq to mean the set of finite sequences $(x_0, x_1, \ldots, x_n)$ of 0's and 1's, and θ for the empty sequence. As usual, 2^ω is the set of infinite sequences of 0's and 1's, endowed with the product topology based on the discrete topology on $\{0, 1\}$. The notions open, closed, $\mathcal{G}_\delta$, Borel, nowhere dense, and first and second category are to be understood as being with respect to this topology. (However, Baire property and meager are not.)

1.1 Category Bases

The definitions in this subsection, and Proposition 1, are due to J. Morgan. (Cf. [2] or [3]).

A *category base* is a pair $(X, \mathcal{C})$ such that X is a nonempty set and $\mathcal{C}$ is a class of subsets of X such that the nonempty sets in $\mathcal{C}$ (the regions) satisfy the axioms:

(1) $X = \bigcup \mathcal{C}$;

(2) Let C be a region, and let $\mathcal{D}$ be a nonempty family of disjoint regions that has power less than the power of $\mathcal{C}$.

   (a) If $C \cap \bigcup \mathcal{D}$ contains a region, then there exists $D \in \mathcal{D}$ such that $C \cap D$ contains a region.

   (b) If $C \cap \bigcup \mathcal{D}$ contains no region, then $C \setminus \bigcup \mathcal{D}$ contains a region.

The most common examples of category bases are (i) all topological spaces, and ii) the pair $(X, \mathcal{C})$, where $\mathcal{C}$ is the class of sets of positive measure with respect to a fixed finite measure on X.

Definition 1 *Let $A \subset X$. A is said to be*

$\mathcal{C}$-singular *if, for every region C there exists a region $C' \subset C$ such that $C' \cap A = \phi$;*

$\mathcal{C}$-meager *if A is a countable union of*

$\mathcal{C}$-singular sets;

$\mathcal{C}$-abundant *if A is not $\mathcal{C}$-meager;*

have the $\mathcal{C}$-Baire property *if, for every region C there exists a region $C' \subset C$ such that either $C' \cap A$ is $\mathcal{C}$-meager or $C' \setminus A$ is $\mathcal{C}$-meager.*

We write $\mathcal{S}(\mathcal{C})$, $\mathcal{M}(\mathcal{C})$, and $\mathcal{B}(\mathcal{C})$, respectively, for the classes of $\mathcal{C}$-singular, $\mathcal{C}$-meager, and $\mathcal{C}$-Baire property sets.

Proposition 1 (Morgan) *Let $(X, \mathcal{C})$ be a category base.*

(i) $\mathcal{M}(\mathcal{C})$ is a σ-ideal on X.

(ii) $\mathcal{B}(\mathcal{C})$ is a σ-algebra on X.

(iii) (The generalized Banach category theorem) Let $A \subset X$. Suppose that for every region C there exists a region $C' \subset C$ such that $C' \cap A$ is meager. Then A is meager.

Now consider the quotient Boolean algebra $\mathcal{P}(X)/\mathcal{M}(\mathcal{C})$, and its important subalgebra, the *category algebra* $\mathcal{B}(\mathcal{C})/\mathcal{M}(\mathcal{C})$. For $A \subset X$, let $[A]$ be the equivalence class of A 'mod $\mathcal{M}(\mathcal{C})$.' We have

Proposition 2 (The generalized Birkhoff-Ulam theorem) *For all category bases $(X,\mathcal{C})$, $\mathcal{B}(\mathcal{C})/\mathcal{M}(\mathcal{C})$ is a complete Boolean algebra, and a regular subalgebra of $\mathcal{P}(X)/\mathcal{M}(\mathcal{C})$.*

In detail: For all $\mathcal{E} \subset \mathcal{B}(\mathcal{C})$, $\sup_{A\in\mathcal{E}}[A]$ exists in the algebra $\mathcal{P}(X)/\mathcal{M}(\mathcal{C})$, and is an element of the algebra $\mathcal{B}(\mathcal{C})/\mathcal{M}(\mathcal{C})$. It follows from the general theory of Boolean algebras that $\sup_{A\in\mathcal{E}}[A]$ also exists in the algebra $\mathcal{B}(\mathcal{C})/\mathcal{M}(\mathcal{C})$, and the two suprema coincide.

For proof see [3], Theorem C15. For the case where $(X,\mathcal{C})$ is a topological space, this is a classic theorem of Birkhoff and Ulam. (Cf. [5], p. 75.)

1.2 A Bit More about Games

We shall require a few more definitions related to the games described in the introduction.

For $K \subset \omega$, we define a K-strategy to be a function τ with domain the set of sequences $(x_0, x_1, \ldots, x_{k-1}) \in Sq$ where $k \in K$, and range $\{0,1\}$. Given a K^c- strategy σ and a K-strategy τ, $\sigma * \tau$ is the element x of 2^ω defined by

$$x_k = \begin{cases} \sigma(x_0, \ldots, x_{k-1}) \text{ if } k \in K^c, \\ \tau(x_0, \ldots, x_{k-1}) \text{ if } k \in K. \end{cases}$$

Let τ be a K-strategy. We define $P(\tau)$, the set of possible outcomes of τ, to be the set $\{\sigma * \tau : \sigma \text{ is a } K^c\text{-strategy }\}$.

Thus, with respect to the game $\Gamma(S, K)$ of the introduction, we have: a winning strategy for player I is a K^c-strategy σ such that $P(\sigma) \subset S$. A winning strategy for player II is a K-strategy τ such that $P(\tau) \cap S = \phi$.

A K-strategy contains much superfluous information, namely the moves which the player using it would make in situations which can never arise during a game in which it is employed. The following proposition may clarify this situation somewhat, and will in any event be useful.

Lemma 3 *Suppose that σ is a K-strategy and σ' is an L-strategy.*

(i) If $\sigma \subset \sigma'$, then $P(\sigma) \supset P(\sigma')$.

(ii) If $P(\sigma) \supset P(\sigma')$, then there exists an L-strategy σ'' such that $\sigma \subset \sigma''$ and $P(\sigma'') = P(\sigma')$. (In particular, $K \subset L$.)

(iii) If $L \setminus K$ is infinite, then $P(\sigma')$ is nowhere dense in $P(\sigma)$, where the latter is endowed with the topology induced as a subspace of 2^ω.

The proof of (i) is entirely straightforward; to prove (ii), let $\sigma''(x_0, \ldots, x_{k-1}) = \sigma(x_0, \ldots, x_{k-1})$ if $k \in K$, and $\sigma'(x_0, \ldots, x_{k-1})$ if $k \in L \setminus K$. To prove (iii), observe that, if $(x_0, \ldots, x_k)$ is any finite sequence in which play has followed the strategy σ, then we may choose $n \in L \setminus K$ such that $n > k$, and extend $(x_0, \ldots, x_k)$ to a sequence $(x_0, \ldots, x_n)$ which still follows σ but not σ', by taking $x_n \neq \sigma'(x_0, \ldots, x_{n-1})$.

We conclude this section with some basic results of Mycielski [1] about the ideal I_M.

Proposition 4 (Mycielski) .

(i) I_M is a translation-invariant σ-ideal on 2^ω.

(ii) I_M contains all singletons (and so, by (i), all countable sets).

(iii) If $S \in I_M$, then there exists a $\mathcal{G}_\delta$ set S' such that $S' \supset S$ and $S' \in I_M$.

2. The Trouble with I_M.

We first prove the negative result mentioned in the introduction.

Theorem 5 *There does not exist a σ-algebra $\mathcal{B}$ on 2^ω which contains the Borel sets and includes I_M, for which $\mathcal{B}/I_M$ is a complete Boolean algebra and a regular subalgebra of $\mathcal{P}(2^\omega)/I_M$.*

PROOF. This is an application of the technique of **21.4** of [5]. It suffices to find a collection of Borel sets with no supremum in the Boolean algebra $\mathcal{P}(2^\omega)/I_M$. To this end, let $(x_\xi : \xi < 2^{\aleph_0})$ be an enumeration of $2^{K_\theta^c}$, and let $(B_\xi : \xi < 2^{\aleph_0})$ be an enumeration of the $\mathcal{G}_\delta$ sets which are elements of I_M. Set $A_\xi = \{x_\xi\} \times 2^{K_\theta}$; thus the sets A_ξ are closed, disjoint, and not elements of I_M. Now the set $\mathcal{E} = \{[A_\xi] : \xi < 2^{\aleph_0}\}$ has no supremum in $\mathcal{P}(2^\omega)/I_M$. Indeed, suppose that A is a subset of 2^ω such that $[A] \geq [A_\xi]$ for $\xi < 2^{\aleph_0}$. Choose $y_\xi \in (A \cap A_\xi) \setminus B_\xi$. Then $Y = \{y_\xi : \xi < 2^{\aleph_0}\} \not\subset B_\eta$ for all η, so by 4(iii), $Y \notin I_M$. On the other hand, $Y \cap A_\xi = \{y_\xi\} \in I_M$, so $[A] > [A - Y] \geq [A_\xi]$, i.e., A is not the supremum of $\mathcal{E}$ in $\mathcal{P}(2^\omega)/I_M$. The proof is complete.

Corollary 6 *There is no category base $(2^\omega, \mathcal{C})$ such that all Borel sets have the $\mathcal{C}$-Baire property, and the class of $\mathcal{C}$-meager sets coincides with I_M.*

3. The Category Base $(2^\omega, C_M^*)$

From here on, we assume the continuum hypothesis.

Recall that, for a strategy σ, $P(\sigma)$ is the set of all possible outcomes of games played according to σ. We define the category base $(2^\omega, \mathcal{C}_M^*)$ by putting

$$\mathcal{C}_M^* = \{P(\sigma) | \sigma \text{ is a } K_s^c\text{-strategy for some } s \in Sq\}.$$

In other words, a region in $\mathcal{C}_M^*$ is the set of possible outcomes of some strategy for player I in one of the games $\Gamma(S, K_s)$.

Remarks:

1. Because we assumed that K_s is infinite for all $s \in Sq$, every region in $\mathcal{C}_M^*$ is a perfect set. In fact, $(2^\omega, \mathcal{C}_M^*)$ is a *perfect base* in the sense of [3].

2. Clearly, if S contains a region, then $S \notin I_M$. A partial converse holds. Call a set S *M-determined* if, for all $s \in Sq$, the game $\Gamma(S, K_s)$ is determined. (In particular, by the theorem of D. A. Martin that all Borel games are determined, every Borel subset of 2^ω is M-determined.) If S is an M-determined set, then S contains a region if, and only if, $S \notin I_M$.

We first show that we in fact have a category base.

Theorem 7 *$(2^\omega, \mathcal{C}_M^*)$ is a category base.*

PROOF. Condition (1) in the definition of category base is obvious. Let C be a region in $\mathcal{C}_M^*$, and let $\mathcal{D}$ be a nonempty family of disjoint regions of power less than the power of $\mathcal{C}_M^*$. Since $\mathcal{C}_M^*$ has the power of the continuum and we have assumed the continuum hypothesis, $\mathcal{D}$ must be countable. (By the way, we shall have occasion to invoke the continuum hypothesis only one other time, in the proof of theorem 13.)

2(i). Suppose that, for all $D \in \mathcal{D}$, $C \cap D$ contains no region. Since each $C \cap D$ is a closed set, by the remark above we have $C \cap D \in I_M$. Since $\mathcal{D}$ is countable and I_M is a σ-ideal, $C \cap \bigcup \mathcal{D} \in I_M$, so $C \cap \bigcup \mathcal{D}$ contains no region.

2(ii). Suppose that $C \cap \bigcup \mathcal{D}$ contains no region. Since $C \cap \bigcup \mathcal{D}$ is a Borel set (an $\mathcal{F}_\sigma$ set, in fact), by another use of the remark above, $C \cap \bigcup \mathcal{D} \in I_M$.

Now suppose for contradiction that $C \setminus \bigcup \mathcal{D}$ contains no region. $C \setminus \bigcup \mathcal{D}$ is a $\mathcal{G}_\delta$ set, and so $C \setminus \bigcup \mathcal{D} \in I_M$. But then $C = (C \cap \bigcup \mathcal{D}) \cup (C \setminus \bigcup \mathcal{D}) \in I_M$, contrary to the hypothesis that C is a region. The proof of the theorem is complete.

Corollary 8 *The Boolean algebra $\mathcal{B}(\mathcal{C}_M^*)/\mathcal{M}(\mathcal{C}_M^*)$ is a complete Boolean algebra, and a regular subalgebra of $\mathcal{P}(X)/\mathcal{M}(\mathcal{C}_M^*)$.*

Lemma 9 *Let $S \subset 2^\omega$. The following are equivalent:*

i) S is $\mathcal{C}_M^*$-singular.

ii) If σ is a K_s^c-strategy, then there exists a $K_{s'}^c$-strategy $\sigma' \supset \sigma$ such that $P(\sigma') \cap S = \phi$.

This follows immediately from 3 and the definition of singular sets.

A category base is called a *Baire base* if no region is meager.

Theorem 10 *$(2^\omega, \mathcal{C}_M^*)$ is a Baire base.*

PROOF. Suppose for contradiction that C is a region, and that $C = \bigcup_{i=0}^{\infty} A_i$, where each A_i is $\mathcal{C}_M^*$-singular. Say $C = P(\sigma)$, where σ is a K_s^c-strategy. Applying 9 repeatedly, we obtain a sequence of strategies $\sigma \subset \sigma_1 \subset \sigma_2 \subset \cdots$, where σ_i is a $K_{s_i}^c$-strategy and $P(\sigma_i) \cap A_i = \phi$. Now consider the $\bigcup_{i=1}^{\infty} K_{s_i}^c$-strategy $\hat{\sigma} = \bigcup_{i=1}^{\infty} \sigma_i$. Then we have, by 3,

$$\phi \neq P(\hat{\sigma}) \subset P(\sigma) \cap \bigcap_{i=1}^{\infty} P(\sigma_i) \subset C \setminus \bigcup_{i=1}^{\infty} A_i$$

which contradicts our initial hypothesis.

We can now characterize the exact relationship between I_M and the $\mathcal{C}_M^*$-meager sets.

Theorem 11 *Let $A \subset 2^\omega$. Then the following are equivalent:*

(i) $A \in I_M$.

(ii) A is M-determined and $\mathcal{C}_M^*$- singular.

(iii) A is M-determined and $\mathcal{C}_M^*$- meager.

PROOF.

(i) $\to$ (ii): Let $A \in I_M$. It is immediate from the definition of I_M that A is M-determined. To see that A is also $\mathcal{C}_M^*$-singular, suppose $s \in Sq$ and σ, a K_s^c-strategy, are given. Let τ be a winning strategy for player II in the game $\Gamma(A, K_{s0})$. Since $K_s^c \cap K_{s0} = \phi$ and $K_{s1}^c \supset K_s^c \cup K_{s0}$, there exists a K_{s1}^c-strategy σ' such that $\sigma' \supset \sigma \cup \tau$. Therefore $P(\sigma') \subset P(\sigma) \cap P(\tau) \subset P(\sigma) \setminus A$, which completes this part of the proof.

(ii) $\to$ (iii) is trivial.

(iii) $\to$ (i): Let $A \subset 2^\omega$ be M-determined and $\mathcal{C}_M^*$-meager. Since $(2^\omega, \mathcal{C}_M^*)$ is a Baire base, A cannot contain a region. By M-determinacy and the remark above, $A \in I_M$.

The proof of 11 is complete.

Corollary 12 *Every I_M-set is $\mathcal{C}_M^*$-meager. Every Borel $\mathcal{C}_M^*$-meager set is in I_M.*

Theorem 13 *The inclusions $I_M \subset \mathcal{S}(\mathcal{C}_M^*) \subset \mathcal{M}(\mathcal{C}_M^*)$ hold and are strict.*

PROOF. The inclusions are given by 11. Now suppose for the purpose of contradiction that $I_M = \mathcal{S}(\mathcal{C}_M^*)$. That makes $\mathcal{S}(\mathcal{C}_M^*)$ a σ-ideal, so in fact $I_M = \mathcal{S}(\mathcal{C}_M^*) = \mathcal{M}(\mathcal{C}_M^*)$. But from 5 and 8, it is clear that $\mathcal{M}(\mathcal{C}_M^*) \neq I_M$, and the first part of the proof is complete.

It remains to show that $\mathcal{S}(\mathcal{C}_M^*) \neq \mathcal{M}(\mathcal{C}_M^*)$. To this end, let $(P(\sigma_\xi) : \xi < 2^{\aleph_0})$ be an enumeration of the $\mathcal{C}_M^*$-regions. We will define by recursion a function $f : 2^{\aleph_0} \to \omega$ and, for $\xi < 2^{\aleph_0}$ and $n \in \omega$, a K_s^c-strategy σ_ξ^n and $x_\xi \in 2^\omega$ such that, for all $\xi, \xi' < 2^{\aleph_0}$, $n \in \omega$,

(1) $\sigma_\xi^n \supset \sigma_\xi$, and σ_ξ^n is a K_s^c-strategy for some $s \in Sq$ of length $\geq n$,

(2) $x_\xi \in P(\sigma_\xi)$

(3) If $\xi' > \xi$, then $x_\xi \notin P(\sigma_{\xi'}^n)$, and

(4) If $\xi' \leq \xi$ and $f(\xi) = n$, then $x_\xi \notin P(\sigma_{\xi'}^n)$.

Assuming for the moment that we have done so, let $A_n = \{x_\xi : f(\xi) = n\}$. Then for all regions $P(\sigma_\xi)$, by (2), $x_\xi \in P(\sigma_\xi) \cap \bigcup_n A_n$, so $\bigcup_n A_n$ is not $\mathcal{C}_M^*$-singular. However, for all $n \in \omega$ and all regions $P(\sigma_\xi)$, by (1), (3) and (4), $P(\sigma_\xi^n) \subset P(\sigma_\xi) \setminus A_n$, so A_n is $\mathcal{C}_M^*$-singular. Thus $\bigcup_n A_n \in \mathcal{M}(\mathcal{C}_M^*) \setminus \mathcal{S}(\mathcal{C}_M^*)$, which completes the PROOF.

To carry out the construction, suppose $\eta < 2^{\aleph_0}$, and $f(\xi)$, σ_ξ^n and x_ξ have been defined for $\xi < \eta$ and $n \in \omega$ so as to satisfy (1)-(4). Say σ_η is a K_s^c-strategy. Since $K_{s0}^c \setminus K_s^c$ is infinite, it is easy to see that there are

$2^{\aleph_0}$ K^c_{s0}-strategies $\sigma \supset \sigma_\eta$ such that the regions $P(\sigma)$ are pairwise disjoint. Temporarily take σ^n, $n \in \omega$, to be any ω of these strategies σ having the additional property that $x_\xi \notin P(\sigma)$ for all $\xi < \eta$. Finally, choose σ^n_η to be some $K^c_{s_n}$ strategy where s_n is of length $\geq n$ and $\sigma^n_\eta \supset \sigma^n$. Thus (1) and (3) hold for $\xi, \xi' \leq \eta$.

Next, choose $\hat{n} \in \omega$ greater than the length of the given sequence s, and let $f(\eta) = \hat{n}$. By 3, $P(\sigma^{\hat{n}}_{\xi'})$ is nowhere dense in $P(\sigma_\eta)$ for $\xi' \leq \eta$. As we are assuming the continuum hypothesis, η is countable, so $\bigcup_{\xi' \leq \eta} P(\sigma^{\hat{n}}_{\xi'})$ is of the first category in $P(\sigma_\eta)$, so we choose $x_\eta \in P(\sigma_\eta) \setminus \bigcup_{\xi' \leq \eta} P(\sigma^{\hat{n}}_{\xi'})$. Thus (2) and (4) are satisfied for $\xi, \xi' \leq \eta$, and the proof of 13 is complete.

Call a set A *strongly M-determined* if, for all closed sets $F \subset 2^\omega$, $A \cap F$ is M-determined. The following guarantees an adequate supply of $\mathcal{C}^*_M$-Baire property sets.

Theorem 14 *Strongly M-determined sets have the $\mathcal{C}^*_M$-Baire property. In particular, Borel sets have the $\mathcal{C}^*_M$-Baire property.*

PROOF. Let A be a strongly M-determined set, and let C be a region. Then C is closed, so $C \cap A$ is M-determined. Thus either $C \cap A$ contains a region or $C \cap A \in I_M$, in which case $C \cap A$ is $\mathcal{C}^*_M$-meager.

Corollary 15 *Analytic sets have the $\mathcal{C}^*_M$-Baire property.*

Indeed, it is shown in [4] that $\mathcal{B}(\mathcal{C})$ is invariant under the operation $(\mathcal{A})$ for all category bases $\mathcal{C}$.

References

[1] J. Mycielski, *Some new ideals of sets on the real line*, Coll. Math. **XX** (1969), 71–76.

[2] J. Morgan, *Baire category from an abstract viewpoint*, Fund. Math. **94** (1977), 13–23.

[3] J. Morgan, *Point set theory*, M. Dekker, New York, 1990.

[4] J. Morgan and K. Schilling, *Invariance under the operation* $(\mathcal{A})$, Proc. Amer. Math. Soc. **100** (1987), 651–654.

[5] R. Sikorski, *Boolean algebras*, Springer-Verlag, N.Y., 1969.