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TRAJECTORY OF THE TURNING POINT IS DENSE FOR ALMOST ALL TENT MAPS

Let $f_a(x) = 1 - a|x|$ for all parameters $a \in (1, 2]$. This is one of the possible models of the family of *tent maps*, that is unimodal maps with constant slope and a maximum at the turning point with this turning point being mapped to the right of itself (or a minimum at the turning point and this turning point being mapped to the left of itself). The parameter a is the slope. Since the topological entropy of f_a is $\log a$ (see [6]), this is also the parametrization by the exponential of topological entropy.

There are other models possible. However, two tent maps with the same slope are conjugate via an affine map. Therefore, as for the family of quadratic maps, the choice of a model does not really matter. We choose this particular model mainly since the turning point is 0 and does not vary with a.

We restrict our attention to the parameters a from $[\sqrt{2}, 2]$. Getting corresponding results for smaller parameter values is very easy. If $\sqrt{2} < a^m \leq 2$ for some $m \in \{1, 2, 2^2, 2^3, \ldots\}$, then the nonwandering set of f_a consists of m disjoint closed intervals and a finite number of periodic points (see e.g. [8], p. 78). Moreover, for such a the map f_a^m restricted to any one of those intervals is a tent map with slope a^m, so it is affinely conjugate to f_{a^m}. This method of dealing with parameter values between 1 and $\sqrt{2}$ is called *renormalization* (see e.g. [2, 3]).

The domain on which we investigate f_a is the smallest invariant interval containing the turning point, that is $[f_a^2(0), f_a(0)]$ (we assume that $a \in [\sqrt{2}, 2]$). On $[f_a^2(0), f_a(0)]$ our map f_a is transitive. We note that $[f_a^2(0), f_a(0)] \cup \{1/(1-a)\}$ is the nonwandering set for f_a with $a \in [\sqrt{2}, 2]$. The only measure we are going to use will be the Lebesgue measure λ. Therefore when we speak about "almost every" and "almost all", we mean this with respect to the Lebesgue measure. Since our maps are not homeomorphisms, the term *trajectory* will always refer to the forward trajectory.

*Presenter

Transitivity of f_a means that the set of points $x \in [f_a^2(0), f_a(0)]$ whose trajectory is dense in $[f_a^2(0), f_a(0)]$ contains a dense G_δ subset of $[f_a^2(0), f_a(0)]$. Since f_a is piecewise expanding, this set also has full Lebesgue measure. This means that generic (in any sense) points of $[f_a^2(0), f_a(0)]$ have dense trajectories. On the other hand, there is one particular point in $[f_a^2(0), f_a(0)]$ whose trajectory is very important. This one point is the turning point. Its trajectory determines the kneading sequence of f_a, and therefore many properties of f_a. This is even more visible for smooth maps conjugate to f_a (see e.g. [2]). Therefore it is important to know how large is the set of parameters for which the turning point has properties shared by generic points. Our main theorem states that this set of parameters has full measure:

Theorem 1 *For almost every $a \in [\sqrt{2}, 2]$ the f_a-trajectory of 0 is dense in $[f_a^2(0), f_a(0)]$.*

By renormalization, it will follow that for almost all $a \in [2^{\frac{1}{2m}}, 2^{\frac{1}{m}}]$ with $m \in \{1, 2, 2^2, 2^3, \ldots\}$, the f_a-trajectory of 0 is dense in the union of the associated m intervals.

One can interpret Theorem 1 in the following way. Let us look at the class of all unimodal maps g for which there exists $a \in [\sqrt{2}, 2]$ such that g is topologically conjugate to f_a. This condition is equivalent to non-renormalizability of g and non-existence of homtervals (see e.g. [2]). For many one-parameter families this is satisfied for a set of parameters of positive measure ([5]). Theorem A says that for this class of maps whenever topological entropy belongs to a certain subset of $[(1/2)\log 2, \log 2]$ of full measure, then the trajectory of the turning point is dense in the interval of transitivity.

It is known (see e.g. [4]) that the set of parameters for which the f_a-trajectory of 0 is not dense in $[f_a^2(0), f_a(0)]$, is dense in $[\sqrt{2}, 2]$, and even more: it is locally uncountable, i.e., its intersection with every open subinterval of $[\sqrt{2}, 2]$ is uncountable.

In [1] it was shown that the f_a-trajectory of 0 is dense in $[f_a^2(0), f_a(0)]$ for a dense G_δ set of values $a \in [\sqrt{2}, 2]$. Thus, our result complements the result in [1]. Other results about tent maps can be found for instance in [4] and in [7]. In [4] it is shown that tent maps have the *shadowing property* (every pseudo-orbit can be approximated by an actual orbit) for almost every $a \in [\sqrt{2}, 2]$; but that the tent maps fail to have the shadowing property for a locally uncountable, dense set of values of a. It is shown also that the shadowing property requires the f_a-trajectory of 0 to contain 0 in its closure. Thus, our result also generalizes some results of [4]. In [7] more general *skew tent maps* are studied. In particular, monotonicity properties of kneading sequences and of topological entropy are established there.

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