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A NOTE ON SYMMETRIC AND ORDINARY DIFFERENTIATION

For a function $f : \mathbb{R} \to \mathbb{R}$ the *symmetric derivative of f at x* is defined as

$$f_s'(x) = \lim_{h \to 0+} \frac{f(x+h) - f(x-h)}{2h},$$

provided this limit exists. We shall adopt the following notation:

$$SD(f) = \{x : f_s'(x) \text{ exists finitely}\},$$

$$D(f) = \{x : f'(x) \text{ exists finitely}\},$$

and

$$C(f) = \{x : f \text{ is continuous at } x\}.$$

That the set difference $SD(f) \setminus D(f)$ is "small" in one sense or another has been established for various classes of functions by a number of authors, *e.g.*, see [1], [3], and [5]. The strongest result of this type for continuous functions has recently been proved by L. Zajíček [6], who has shown the following:

Theorem Z *If $f : \mathbb{R} \to \mathbb{R}$ and $C(f) = \mathbb{R}$, then the set $SD(f) \setminus D(f)$ is σ–symmetrically porous. More specifically, for each $0 < \epsilon < 1$ this set is σ–$(1-\epsilon)$–symmetrically porous.*

Some terminology needs to be reviewed in order to appreciate what this theorem is saying. If A is a subset of the real line $\mathbb{R}$ and $x \in \mathbb{R}$, then the *porosity of A at x* is defined to be

$$\limsup_{r \to 0+} \frac{\lambda(A, x, r)}{r},$$

where $\lambda(A,x,r)$ is the length of the longest open interval contained in either $(x,x+r)\cap A^c$ or $(x-r,x)\cap A^c$ and A^c denotes the complement of A. A set is said to be *porous at* x if it has positive porosity at x and is called a *porous set* if it is porous at each of its points. Further, a set is called σ*-porous* if it is a countable union of porous sets. The *symmetric porosity of* A *at* x is defined as

$$\limsup_{r\to 0+}\frac{\gamma(A,x,r)}{r},$$

where $\gamma(A,x,r)$ is the supremum of all positive numbers h such that there is a positive number t with $t+h\le r$ such that both of the intervals $(x-t-h,x-t)$ and $(x+t,x+t+h)$ lie in A^c. A set A is *symmetrically porous* if it has positive symmetric porosity at each of its points. For a number $0<\alpha\le 1$ the set A is called α*–symmetrically porous* if it has symmetric porosity at least α at each of its points. The set A is called σ*–symmetrically porous* if it is a countable union of symmetrically porous sets, and is called σ*–*α*–symmetrically porous* if it is a countable union of α-symmetrically porous sets.

The observations that σ–porous sets are not necessarily σ–symmetrically porous and that σ–symmetrically porous sets need not be σ–$(1-\epsilon)$ symmetrically porous, for every $0<\epsilon<1$, have been established in [2] and [4].

At the Sixteenth Summer Symposium on Real Analysis in Smolenice, Czechoslovakia, in August 1991, Zajíček posed the problem of whether or not Theorem Z would remain valid if the assumption that $C(f)=\mathbb{R}$ were replaced by the assumption that $\overline{C(f)}=\mathbb{R}$, where $\overline{C(f)}$ denotes the closure of $C(f)$. The purpose of this note is to show that Theorem Z does, indeed, remain valid under that modification in hypothesis. In fact, we establish a slightly stronger result (Theorem 1). We shall show that it suffices to assume that $SD(f)\subseteq\overline{C(f)}$.

Before moving to the proof of our result, it should be noted that a careful reading of [6] shows that Zajíček's proof actually yields a more general result than that stated in Theorem Z. Indeed, his proof shows the following:

Theorem Z* *For a function* $f:\mathbb{R}\to\mathbb{R}$, *the set* $[SD(f)\cap C(f)]\setminus D(f)$ *is* σ*-*$(1-\epsilon)$*-symmetrically porous for each* $0<\epsilon<1$.

Thus, to establish our result, it will suffice to focus our attention on the set $SD(f) \setminus C(f)$. This is accomplished in the following proposition. The basic idea of the proof remains the same as that utilized by Zajíček, except that where he reflected points, we shall reflect intervals.

Proposition 1 *For* $f : \mathbb{R} \to \mathbb{R}$ *let*

$$\mathcal{S}(f) = \left\{ x : \limsup_{h \to 0+} \left| \frac{f(x+h) - f(x-h)}{2h} \right| < +\infty \right\}.$$

Then $(\mathcal{S}(f) \cap \overline{C(f)}) \setminus C(f)$ *is* σ*-1-symmetrically porous.*

Proof. Let $T = (\mathcal{S}(f) \cap \overline{C(f)}) \setminus C(f)$, and for all natural numbers m, n, and l, let

$$T_m = \left\{ x \in T : \limsup_{h \to 0+} \left| \frac{f(x+h) - f(x-h)}{2h} \right| < m \right\},$$

$$T_{m,n} = \left\{ x \in T_m : \left| \frac{f(x+h) - f(x-h)}{2h} \right| < m \text{ for all } 0 < h < \frac{1}{n} \right\},$$

and

$$T_{m,n,l} = \left\{ x \in T_{m,n} : \omega_f(x) > \frac{1}{l} \right\},$$

where $\omega_f(x)$ denotes the oscillation of f at x. Since

$$T = \bigcup_{m,n,l} T_{m,n,l}$$

it will suffice to show that each set $T_{m,n,l}$ is 1–symmetrically porous.

Suppose to the contrary that there are natural numbers m, n, and l such that $T_{m,n,l}$ is not 1–symmetrically porous and let $x_0 \in T_{m,n,l}$ be a point at which $T_{m,n,l}$ has symmetric porosity less than 1. Choose a positive number ϵ such that the symmetric porosity of $T_{m,n,l}$ at x_0 is less than $1 - \epsilon$. Without loss of generality we may assume that $x_0 = 0$. Since $T_{m,n,l}$ is not $(1-\epsilon)$–symmetrically porous at 0, there is a number $0 < \eta < 1/n$ such that for each $0 < t < \eta$

$$[(-t + \epsilon t/2, -\epsilon t/2) \cup (\epsilon t/2, t - \epsilon t/2)] \cap T_{n,m,l} \neq \emptyset. \qquad (1)$$

Let

$$B = \frac{16m}{\epsilon}. \tag{2}$$

Since 0 is a limit point of $C(f)$, we select a point $a_1 \in C(f)$ such that

$$|a_1| < \min\left\{\eta, \frac{1}{2lB}\right\}. \tag{3}$$

We shall now inductively define a sequence $\{a_k\}$ of points such that for each natural number k we have

$$|a_{k+1}| \le (1-\epsilon)|a_k|. \tag{4}$$

If the points $a_1, \ldots, a_k$ have been selected we proceed as follows. First, if $a_k = 0$, we put $a_{k+1} = 0$. If $a_k \neq 0$ then from (1) it follows that there is a point

$$p_k \in [(-|a_k| + \epsilon|a_k|/2, -\epsilon|a_k|/2) \cup (\epsilon|a_k|/2, |a_k| - \epsilon|a_k|/2)] \cap T_{m,n,l}\ , \tag{5}$$

where for definiteness we shall choose p_k to have the same sign as a_k if we have a choice. We then set

$$a_{k+1} = \begin{cases} 2p_k - a_k & \text{if } p_k a_k > 0 \\ 2p_k + a_k & \text{if } p_k a_k < 0 \end{cases}. \tag{6}$$

In other words, a_{k+1} is the reflection of a_k about p_k if a_k and p_k are on the same side of the origin; otherwise, a_{k+1} is the reflection of $-a_k$ about p_k. From (5) and (6) inequality (4) clearly follows for each natural number k.

Since $a_1 \in C(f)$, there is a $0 < \delta < \eta - |a_1|$ such that

$$|f(x) - f(a_1)| < \frac{1}{4l} \text{ for all } x \in I_1 \equiv (a_1 - \delta, a_1 + \delta). \tag{7}$$

Since $\omega_f(0) > 1/l$, (7) implies that $0 \notin I_1$, *i.e.*,

$$\delta \le |a_1|. \tag{8}$$

For each k let $I_k = (a_k - \delta, a_k + \delta)$. Since $\lim_{k\to\infty} a_k = 0$, there is a smallest natural number N for which $0 \in I_{N+1}$. For each $k = 1, \ldots, N$ define a function $r_k : \mathbb{R} \to \mathbb{R}$ by

$$r_k(x) = \begin{cases} 2p_k - x & \text{if } p_k x > 0 \\ 2p_k + x & \text{if } p_k x \le 0 \end{cases}. \tag{9}$$

Consequently, $r_k(x)$ is the reflection of x about p_k if x and p_k are on the same side of the origin; otherwise, $r_k(x)$ is the reflection of $-x$ about p_k. In this notation,

$$a_{k+1} = r_k(a_k) \text{ and } I_{k+1} = r_k(I_k) \text{ for all } k = 1, \dots, N,$$

the latter resulting from the fact that for $k = 1, \dots, N$, we must have $\delta \leq |a_k|$, and hence for all $x \in I_k$, $p_k x > 0$ if and only if $p_k a_k > 0$.

Thus, for each $s \in I_{N+1}$ there are points $s_N \in I_N$, $s_{N-1} \in I_{N-1}$, $\dots$, $s_1 \in I_1$ such that

$$s = r_N(s_N), s_N = r_{N-1}(s_{N-1}), \dots, s_2 = r_1(s_1).$$

We shall observe that

$$|f(r_j(s_j)) - f(s_j)| < \frac{B\epsilon}{2}|a_j| \text{ for each } j = 1, 2, \dots, N. \tag{10}$$

To see this, fix a $j \in \{1, 2, \dots, N\}$. Either $s_j p_j > 0$ or $s_j p_j < 0$. Let's consider the former situation first. Then

$$\begin{aligned} |f(r_j(s_j)) - f(s_j)| &= |f(2p_j - s_j) - f(s_j)| \\ &\leq 2|s_j - p_j|m \\ &< 2(|a_j| + \delta)m \\ &\leq 4|a_j|m \\ &= \frac{B\epsilon}{4}|a_j|, \end{aligned} \tag{11}$$

where we have used the fact that $p_j \in T_{m,n,l}$. Next, if $s_j p_j < 0$, then

$$\begin{aligned} |f(r_j(s_j)) - f(s_j)| &= |f(2p_j + s_j) - f(s_j)| \\ &\leq |f(2p_j + s_j) - f(-s_j)| + |f(-s_j) - f(s_j)| \\ &< 2|-s_j - p_j|m + 2|s_j|m \\ &< 4(|a_j| + \delta)m \\ &\leq 8|a_j|m \\ &= \frac{B\epsilon}{2}|a_j|. \end{aligned} \tag{12}$$

This time we used the fact that both 0 and p_j are in $T_{m,n,l}$. From (11) and (12) we obtain the claim (10).

Hence, for each $s \in I_{N+1}$ we have

$$\begin{aligned}
|f(s)-f(s_1)| &= |f(r_N(s_N))-f(s_1)| \\
&\le \sum_{j=1}^{N}|f(r_j(s_j))-f(s_j)| \\
&< \sum_{j=1}^{N}\frac{B\epsilon}{2}|a_j| \qquad \text{(from (10))} \\
&\le \frac{B\epsilon}{2}\sum_{j=1}^{N}(1-\epsilon)^{j-1}|a_1| \qquad \text{(from (4))} \\
&< \frac{B\epsilon|a_1|}{2}\sum_{j=1}^{\infty}(1-\epsilon)^{j-1} \\
&= \frac{B|a_1|}{2} \\
&< \frac{1}{4l}, \qquad (13)
\end{aligned}$$

where the last inequality follows from (3). So, for each $s \in I_{N+1}$ we have

$$\begin{aligned}
|f(s)-f(a_1)| &\le |f(s)-f(s_1)|+|f(s_1)-f(a_1)| \\
&< \frac{1}{4l}+\frac{1}{4l} \qquad \text{(from (13) and (7))} \\
&= \frac{1}{2l}.
\end{aligned}$$

This is an impossible situation, however, since $0 \in I_{N+1}$ and $\omega_f(0) > 1/l$. This contradiction completes the proof of the proposition.

Theorem 1 *Let $f : \mathbb{R} \to \mathbb{R}$. If $SD(f) \subseteq \overline{C(f)}$, then $SD(f) \setminus D(f)$ is σ-$(1-\epsilon)$-symmetrically porous for each $0 < \epsilon < 1$.*

Proof. We have

$$SD(f) \setminus D(f) = [(SD(f) \cap C(f)) \setminus D(f)] \cup [SD(f) \setminus C(f)].$$

Let $0 < \epsilon < 1$. According to Theorem Z*, the set $[SD(f) \cap C(f)] \setminus D(f)$ is σ-$(1-\epsilon)$-symmetrically porous, and according to Proposition 1, the set $SD(f) \setminus C(f)$ is σ-1-symmetrically porous. Consequently, the set $SD(f) \setminus D(f)$ is σ-$(1-\epsilon)$-symmetrically porous.

References

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