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Washek F. Pfeffer, Department of Mathematics, University of California, Davis, CA 95616.

Wei-Chi Yang, Department of Mathematics and Statistics, Radford University, Redford, VA 24142.

A NOTE ON CONDITIONALLY CONVERGENT INTEGRALS

Recently, S.P. Lu and P.Y. Lee obtained an interesting characterization of the Henstock-Kurzweil integrability based on a suitable truncation of integrable functions (see [**LL**]). We generalize their result to an abstract setting that encompasses all the Riemann type integrals found in the present day literature.

Throughout this note, $(X, \mathfrak{M}, \mu)$ is a fixed measure space with a *finite* and *nonatomic* measure μ. All functions we consider are real-valued functions defined on measurable subsets of X. The Lebesgue integral of a function f on a set $E \in \mathfrak{M}$ with respect to μ is denoted by $\int_E f$. If $E \in \mathfrak{M}$, then $L^1(E)$ is the linear space of all measurable functions f on E for which $\int_E |f| < +\infty$. For each $f \in L^1(X)$, we set $|f|_1 = \int_X |f|$.

A *partition* is a collection (possibly empty) $P = \{(\theta_1, x_1), \ldots, (\theta_p, x_p)\}$ where $x_1, \ldots, x_p$ are points of X, and $\theta_1, \ldots, \theta_p$ are nonnegative functions from $L^1(X)$ such that $\sum_{i=1}^p \theta_i \leq 1$ almost everywhere in X. If f is a function on X and $E \subset X$, we let

$$\sigma(f, P; E) = \sum \{f(x)|\theta|_1 : (\theta, x) \in P \text{ and } x \in E\}.$$

We write $\sigma(f, P)$ instead of $\sigma(f, P; X)$; thus $\sigma(f, P) = \sum_{i=1}^p f(x_i)|\theta_i|_1$.

Remark 1. Our general concept of a partition includes the usual partitions used in [**LL**], the approximating partitions of [**P**], and also the partitions of unity employed in [**JK**] and [**KMP**].

In the family of all partitions, we fix once and for all a *filter base* $\mathfrak{P}$, i.e., a collection of nonempty families Π of partitions such tha for each Π_1 and Π_2 in $\mathfrak{P}$ there is a Π in $\mathfrak{P}$ with $\Pi \subset \Pi_1 \cap \Pi_2$.

Definition 2. A function f on X is called *integrable* if there is a real number I having the following property: given $\varepsilon > 0$, we can find a $\Pi \in \mathfrak{P}$ so that

$$|\sigma(f, P) - I| < \varepsilon$$

for each $P \in \Pi$.

Since $\mathfrak{P}$ is a filter base, the number I of Definition 2 is determined uniquely by the integrable function f; it is called the *integral* of f denoted by $\int^* f$.

It is easy to verify that the family $\mathcal{I}$ of all integrable functions is a linear space and that the map $f \mapsto \int^* f$ is a nonnegative linear functional on $\mathcal{I}$. A simple proof of the following Cauchy test for integrability is also left to the reader.

Lemma 3. *A function f on X is integrable if and only if given $\varepsilon > 0$, there is a $\Pi \in \mathfrak{P}$ such that*

$$|\sigma(f, P) - \sigma(f, Q)| < \varepsilon$$

for each P and Q in Π.

It is clear that without additional hypotheses little can be said about the integral $\int^*$. Thus we make two natural assumptions that are easy to verify for each available version of the Henstock-Kurzweil integral.

Assumptions 4. Throughout, we shall assume that the following conditions are satisfied:

(i) $L^1(X) \subset \mathcal{I}$, *and* $\int^* f = \int_X f$ *for each* $f \in L^1(X)$;
(ii) *each* $f \in \mathcal{I}$ *is measurable.*

Proposition 5. *The following statements are true.*

(1) *If f and g are functions on X and $f = g$ almost everywhere, then $f \in \mathcal{I}$ if and only if $g \in \mathcal{I}$, in which case $\int^* f = \int^* g$.*
(2) *A function f on X belongs to $L^1(X)$ if and only if both f and $|f|$ belong to $\mathcal{I}$.*
(3) *If $\{f_n\}$ is a sequence in $\mathcal{I}$ converging pointwise to a function f on X, then $f \in \mathcal{I}$ whenever either of the following conditions holds:*
   (a) *$f_1 \le f_2 \le \cdots$ and $\sup \int^* f_n < +\infty$;*
   (b) *$g \le f_n \le h$ for some g and h in $\mathcal{I}$ and $n = 1, 2, \ldots$.*

Proof. We prove each assertion separately.

(1) Clearly, $f - g \in L^1(X)$ and $\int_X (f - g) = 0$. Thus if $g \in \mathcal{I}$, it follows from 4(i) that $f = (f - g) + g$ belongs to $\mathcal{I}$ and $\int^* f = \int^* g$. The proof is completed by symmetry.
(2) If f and $|f|$ belong to $\mathcal{I}$, then $f_n = \min\{|f|, n\}$, $n = 1, 2, \ldots$, belong to $L^1(X)$ by 4(ii). As $f_n \nearrow |f|$, it follows from 4(i) that

$$\int_X |f| = \lim \int_X f_n = \lim \int^* f_n \le \int^* |f| < +\infty .$$

The converse is a direct consequence of 4(i).
(3) Using (2) and 4(i), it suffices to apply the monotone and dominated convergence theorems for the Lebesgue integral to the sequences $\{f_n - f_1\}$ and $\{f_n - g\}$, respectively.

So far we have not used the fact that the measure μ is nonatomic; however, the next lemma depends on it in an essential way

Lemma 6. *Let $A \subset B$ be measurable sets and let $f \in L^1(B)$. If $\int_A f \le c \le \int_B f$, then there is a measurable set C such that $A \subset C \subset B$ and $\int_C f = c$.*

Proof. On the σ-algebra $\mathfrak{N} = \{E \in \mathfrak{M} : E \subset B - A\}$ consider the signed measure $\nu : E \mapsto \int_E f$. Since μ is a nonatomic measure, it is easy to verify that so is ν. According to Liapounoff's theorem ([**R**, Theorem 5.5]), the set $\{\nu(E) : E \in \mathfrak{N}\}$ is a compact interval. As

$$\nu(\emptyset) = 0 \le c - \int_A f \le \nu(B - A),$$

there is a $D \in \mathfrak{N}$ with

$$\int_D f = \nu(D) = c - \int_A f.$$

Consequently, $C = A \cup D$ is the desired set.

Proposition 7. *Let f be a measurable function with $\int_X f^+ = \int_X f^- = +\infty$, and let c be a real number. There is an increasing sequence $\{X_n\}$ of measurable sets whose union is X and such that $f \in L^1(X_n)$ and $\int_{X_n} f = c$ for $n = 1, 2, \ldots$.*

Proof. We follow the proof of [**LL**, Lemma 2]. For $n = 1, 2, \ldots$, let

$A_n^+ = \{x \in X : n-1 \le f(x) < n\}$ and $A_n^- = \{x \in X : -n \le f(x) < -(n-1)\}$.

Since

$$\sum_{n=1}^{\infty} \int_{A_n^+} f = \int_X f^+ = +\infty \qquad \text{and} \qquad \sum_{n=1}^{\infty} \int_{A_n^-} f = -\int_X f^- = -\infty,$$

we can construct inductively increasing sequences $\{r_k\}$ and $\{s_k\}$ of positive integers so that

$$\sum_{n=1}^{r_k} \int_{A_n^+} f + \sum_{n=1}^{s_k} \int_{A_n^-} f < c < \sum_{n=1}^{r_{k+1}} \int_{A_n^+} f + \sum_{n=1}^{s_k} \int_{A_n^-} f$$

for $k = 1, 2, \ldots$. Thus letting

$$E_k^- = \left(\bigcup_{n=1}^{r_k} A_n^+ \right) \cup \left(\bigcup_{n=1}^{s_k} A_n^- \right) \qquad \text{and} \qquad E_k^+ = \left(\bigcup_{n=1}^{r_{k+1}} A_n^+ \right) \cup \left(\bigcup_{n=1}^{s_k} A_n^- \right),$$

we have $E_1^- \subset E_1^+ \subset E_2^- \subset E_2^+ \subset \cdots$,

$$\bigcup_{k=1}^{\infty} E_k^- = \bigcup_{k=1}^{\infty} E_k^+ = X, \quad \text{and} \quad \int_{E_k^-} f < c < \int_{E_k^+} f \quad \text{for} \quad k = 1, 2, \ldots.$$

By Lemma 6, there are measurable sets X_k such that $E_k^- \subset X_k \subset E_k^+$ and $\int_{X_k} f = c$. Since $E_k^- \subset X_k \subset E_k^+ \subset E_{k+1}^- \subset X_{k+1}$, we see that $\{X_k\}$ is the desired sequence.

Corollary 8. *If $f \in \mathcal{I}$, then there is an increasing sequence $\{X_n\}$ of measurable sets whose union is X and such that $f \in L^1(X_n)$ and $\int_{X_n} f = \int^* f$ for $n = 1, 2, \ldots$.*

Proof. If $f \in L^1(X)$, then in view of 4(i), it suffices to let $X_n = X$ for $n = 1, 2, \ldots$. If f does not belong to $L^1(X)$, then by Proposition 5(2), neither does f^+ nor f^-, and the corollary follows from Proposition 7.

Remark 9. Propositions 5 and 7, as well as Corollary 8, are true for any linear space $\mathcal{I}$ and any nonnegative linear functional $f \mapsto \int^* f$ on $\mathcal{I}$ that satisfy both conditions of Assumption 4. The approximation of $\int^* f$ by the sums $\sigma(f, P)$ was not used in obtaining these results.

Theorem 10. *A function f on X belongs to $\mathcal{I}$ if and only if given $\varepsilon > 0$, there are $E \in \mathfrak{M}$ and $\Pi \in \mathfrak{P}$ such that $f \in L^1(E)$ and*

$$|\sigma(f, P; X - E)| < \varepsilon$$

for each $P \in \Pi$. If $f \in \mathcal{I}$, then E can be chosen so that $\mu(X - E) < \varepsilon$ and $\int_E f = \int^* f$.

Proof. Choose an $\varepsilon > 0$, and suppose first that there are $E \in \mathfrak{M}$ and $\Pi \in \mathfrak{P}$ such that $f \in L^1(E)$ and $|\sigma(f, P; X - E)| < \varepsilon/3$ for each $P \in \Pi$. Define a function $g \in L^1(X)$ by setting

$$g(x) = \begin{cases} f(x) & \text{if } x \in E, \\ 0 & \text{if } x \in X - E. \end{cases}$$

According to 4(i) and Lemma 3, there is a $\Pi_g \in \mathfrak{P}$ such that $|\sigma(g, P) - \sigma(g, Q)| < \varepsilon/3$ for each P and Q in Π_g. Now if $\Pi_f \in \mathfrak{P}$ and $\Pi_f \subset \Pi \cap \Pi_g$, then

$$|\sigma(f, P) - \sigma(f, Q)| \le |\sigma(g, P) - \sigma(g, Q)| + |\sigma(f, P; X - E)| + |\sigma(f, Q; X - E)| < \varepsilon$$

for every P and Q in Π_f. Consequently, $f \in \mathcal{I}$ by Lemma 3.

Conversely, suppose that $f \in \mathcal{I}$ and find $\Pi_f \in \mathfrak{P}$ so that $|\sigma(f, P) - \int^* f| < \varepsilon/2$ for each $P \in \Pi_f$. By Corollary 8, there is an $E \in \mathfrak{M}$ such that $\mu(X - E) < \varepsilon$, $f \in L^1(E)$, and $\int_E f = \int^* f$. Again, define a function $g \in L^1(X)$ by setting

$$g(x) = \begin{cases} f(x) & \text{if } x \in E, \\ 0 & \text{if } x \in X - E. \end{cases}$$

According to 4(i), the function g belongs to $\mathcal{I}$ and

$$\int^* g = \int_X g = \int_E f = \int^* f.$$

There is a $\Pi_g \in \mathfrak{P}$ such that $|\sigma(g, P) - \int^* g| < \varepsilon/2$ for each $P \in \Pi_g$. Thus if $\Pi \in \mathfrak{P}$ and $\Pi \subset \Pi_f \cap \Pi_g$, then

$$|\sigma(f, P; X - E)| = |\sigma(f, P) - \sigma(g, P)| \le \left|\sigma(f, P) - \int^* f\right| + \left|\sigma(g, P) - \int^* g\right| < \varepsilon$$

for each $P \in \Pi$.

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