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Some theorems whose σ–porous exceptional sets are not σ–symmetrically porous

During the past couple of decades a number of results fitting the following format have been established: *A function $f : \mathbb{R} \to \mathbb{R}$ has some property $\mathcal{P}$ at every point except those in a σ-porous set.* The following four theorems are representative.

Theorem 1 [10] *For an arbitrary function $f : \mathbb{R} \to \mathbb{R}$ the left essential cluster set of f at x equals the right essential cluster set of f at x except for those x in a σ-porous set.*

Theorem 2 [4] *If $f : \mathbb{R} \to \mathbb{R}$ is monotone, then the left upper derivate of f at x equals the right upper derivate of f at x except for those x in a σ-porous set.*

Theorem 3 [2] *If $f : \mathbb{R} \to \mathbb{R}$ has a dense set of points of continuity, then the upper symmetric derivate of f at x equals the maximum of the left and right upper derivates of f at x except for those x in a σ-porous set.*

Theorem 4 [1] *For an arbitrary function $f : \mathbb{R} \to \mathbb{R}$, the set of points at which f has a finite right derivative, but not a two sided derivative, is a σ-porous set.*

L. Zajíček [11] has recently shown that if we restrict ourselves to continuous functions and replace right differentiability in Theorem 4 by symmetric differentiability, then a different conclusion can be reached:

Theorem 5 [11] *For a continuous function $f : \mathbb{R} \to \mathbb{R}$, the set of points at which f has a finite symmetric derivative, but not a two sided derivative, is a σ-symmetrically porous set.*

While σ–symmetrically porous sets are necessarily σ–porous, the existence of σ–porous sets which are not σ–symmetrically porous has been established in [5] and [8]. Thus Zajíček's result is not simply an application of Theorem 3. However, a natural question seems to be "Can the exceptional set in Theorem 3 be shown to be σ–symmetrically porous? If not in general, then perhaps under the assumption of continuity?" More generally, "Can any or all of the exceptional sets in Theorems 1 - 4 be shown to be σ–symmetrically porous?" The purpose of this note is to show that the answer to this general question is 'no', even if we are looking at fairly 'nice' functions. This will be accomplished with basically one construction in Proposition 1.

We begin by reviewing the pertinent definitions and notation. We shall use $|I|$ to denote the length of an interval I. If A is a subset of the real line $\mathbb{R}$ and $x \in \mathbb{R}$, then the *porosity of A at x* is defined to be

$$\limsup_{r \to 0+} \frac{\lambda(A, x, r)}{r},$$

where $\lambda(A, x, r)$ is the length of the longest open interval contained in either $(x, x+r) \cap A^c$ or $(x-r, x) \cap A^c$ and A^c denotes the complement of A. A set is said to be *porous at x* if it has positive porosity at x and is called a *porous set* if it is porous at each of its points and a set is called *σ-porous* if it is a countable union of porous sets. The *symmetric porosity of A at x* is defined as

$$\limsup_{r \to 0+} \frac{\gamma(A, x, r)}{r},$$

where $\gamma(A, x, r)$ is the supremum of all positive numbers h such that there is a positive number t with $t + h \leq r$ such that both of the intervals $(x - t - h, x - t)$ and $(x + t, x + t + h)$ lie in A^c. A set A is *symmetrically porous* if it has positive symmetric porosity at each of its points and is called *σ-symmetrically porous* if it is a countable union of symmetrically porous sets. More information on symmetric porosity and some uses can be found in [3], [5], [8], [9], and [11].

For a function $f : \mathbb{R} \to \mathbb{R}$ the *left [right] essential cluster set of f at x* is the set of all numbers y for which $f^{-1}(U)$ has positive left [right] upper outer density at x for each neighborhood U of y. The *upper symmetric derivate of f at x* is

$$\limsup_{h \to 0+} \frac{f(x+h) - f(x-h)}{2h},$$

while the upper left and right derivates of f at x have their usual meanings. Following O'Malley [7], we shall say that a function is a *Baire* 1* function if each perfect set contains a portion (*i.e.*, a nonempty intersection of an open set with the perfect set) on which the restriction of the function is continuous.

The following proposition shows that none of the exceptional sets in Theorems 1–4 need be σ–symmetrically porous. More precisely, it shows that this is the case for Theorem 1 even if the function is Baire* 1 and Darboux, and that this is the case in Theorems 2–4 even if the function is monotone and Lipschitz.

Proposition 1 *There is a non-σ-symmetrically porous set $A \subset [0,1]$, a Baire* 1, Darboux function $f : \mathbb{R} \to \mathbb{R}$, and a monotone Lipschitz function $g : \mathbb{R} \to \mathbb{R}$ such that for each $x \in A$ we have*

1. *The left essential cluster set of f at x does not equal the right essential cluster set of f at x.*

2. *The left derivative of g at x exists and equals 0, while the right upper derivate of g at x equals 1, and, consequently, the upper symmetric derivate of g at x equals $1/2$.*

Proof. First, we shall define a certain symmetric Cantor set C. Let $\{\alpha_n\}$ be a sequence of positive numbers less than one having limit zero. We define C as the intersection of the closed sets E_n, where $E_0 \equiv [0,1]$, and for $n \geq 1$, E_n is the union of the 2^n disjoint closed intervals obtained by deleting from each interval J in collection of 2^{n-1} disjoint closed intervals whose union is E_{n-1} the open interval centered in J whose length is $\alpha_n|J|$.

Corollary 1 in [5] indicates that C has symmetric porosity zero at each of its points. (For what we do here we really don't care whether C has positive measure or measure zero; but it is perhaps worth noting that we could take C to have positive measure by further requiring that the sequence $\{\alpha_n\}$ be summable [6].) Furthermore, no set S which is residual in C can be σ–symmetrically porous. This was essentially shown in the proof of Theorem 1 in [5], but since it was not stated in quite this generality and since the proof is short, here is the argument again.

Suppose that S is a residual subset of C and that S is σ-symmetrically porous, say

$$S = \bigcup_{n=1}^{\infty} S_n,$$

where each S_n is symmetrically porous. Since S is residual in C, there would exist an open interval I and a natural number n_0 such that $I \cap C$ is nonempty and S_{n_0} is dense in $I \cap C$. However, this leads to a contradiction, for if $x \in S_{n_0} \cap I \cap C$, then S_{n_0} has positive symmetric porosity at x, while C has symmetric porosity 0 at x, an impossible situation due to the denseness of S_{n_0} in $C \cap I$.

The set A that we seek in the present proposition will be a certain residual subset of C.

Enumerate the contiguous intervals to C in a sequence $\{I_m = (a_m, b_m)\}$. For each natural number m let

$$A_m = \left(a_m - \frac{1}{m^2}|I_m|, a_m\right),$$

and

$$B_m = \left(a_m, a_m + \frac{1}{m}|I_m|\right).$$

Then set

$$A = \left(\bigcap_{j=1}^{\infty} \bigcup_{m=j}^{\infty} A_m\right) \cap C,$$

and

$$B = \bigcup_{m=1}^{\infty} B_m.$$

Note that for each j, $\bigcup_{m=j}^{\infty} A_m \cap C$ is clearly a dense $\mathcal{G}_\delta$ subset of C, and, hence, so is A. Consequently, A is residual in C and is therefore not σ–symmetrically porous.

Since the relative measure of B in each subinterval of the form (c, b_m) of each contiguous interval (a_m, b_m) is at most $1/m$, an easy computation shows that the left density of B at each point of C is zero. In particular, this is true at each point in A. Further, the upper right density of B at each $x \in A$ is clearly one, since x belongs to infinitely many A_m.

Let $h : \mathbb{R} \to \mathbb{R}$ denote the characteristic function of the open set B. Then at each $x \in A$ we have that the left essential cluster set of h at x is $\{0\}$, while the right essential cluster set at x is $\{0,1\}$. This function h is clearly a Baire* 1 function, but lacks the Darboux property. We may modify it slightly to obtain the Baire* 1, Darboux function f mentioned in the statement of the proposition by proceeding in the following manner: On each interval $B_m = (a_m, c_m)$ let f linearly increase from 0 to 1 on $(a_m, a_m + |B_m|/m]$, be constantly 1 on $[a_m + |B_m|/m, c_m - |B_m|/m]$, and linearly decrease from 1 to 0 on $[c_m - |B_m|/m, c_m)$. Let f be 0 on the complement of B. Then at each $x \in A$ the left essential cluster set of f at x is $\{0\}$, while the right essential cluster set of f at x is $\{0,1\}$.

The monotone Lipschitz function g in the statement of the proposition can then be taken to be

$$g(x) = \int_{(-\infty,x]} h,$$

for in that event, we have that for each $x \in A$, the left derivative of g is simply the left density of B, *i.e.* 0, while the right upper derivate of g at such an x is the right upper density of B at x, namely 1.

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