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ON SAKS-HENSTOCK LEMMA FOR THE RIEMANN-TYPE INTEGRALS

The first key step in the theory of the Henstock integral is Henstock's lemma, or what Henstock called the Saks-Henstock lemma (see, for example, [2; p.12]). As we can see, the lemma makes easy many proofs of the theorems later on. If we consider the McShane integral [2,3] and mimic the proof of the lemma, we obtain the Saks-Henstock lemma for the McShane integral. Then the monotone convergence theorem using the lemma will have an easier proof for the McShane integral than otherwise (see [3; p.86]). This observation leads us to conjecture that for the Riemann-type integrals it is easier to prove results using the corresponding Saks-Henstock lemma. In this paper, we elaborate our point by making use of the RL integral, a Riemann-type integral as defined in [1,2]. The motivation for considering the RL integral is computational. It is well-known that the Riemann integral provides a convenient way of computing the integral value. But the Lebesgue integral does not, and nor does the McShane integral because of unequally spaced divisions. Note that the RL integral uses equally spaced divisions which are useful in computation. We assume that the reader has some familarity with the Henstock integral.

We recall that a non-negative function f is said to be RL integrable on $[a,b]$ if there exists a number A such that for every $\varepsilon > 0$ and $\eta > 0$ there exist an open set G and a constant $\delta > 0$ such that $|G| < \eta$ and that for every division $D = \{([u,v],\xi)\}$ with $0 < v-u < \delta$ and $\xi \in [u,v]$ we have

$$|(D)\sum_{\xi \notin G} f(\xi)(v-u) - A| < \varepsilon,$$

where the sum is taken over all $([u,v],\xi)$ in D in which $\xi \in [u,v]-G$. In other words, the term $f(\xi)(v-u)$ is not included in the above sum when $[u,v]-G$

is empty, and when $[u, v] - G$ is nonempty we always take $\xi \in [u, v] - G$. For notation, see [2].

In general, a function f is said to be RL integrable on $[a, b]$ if both $f^+ = max(f, 0)$ and $f^- = max(-f, 0)$ are RL integrable on $[a, b]$, and the integral of f is that of $f^+ - f^-$. It is known [2; p.105] that the RL integral is equivalent to the absolute Henstock integral, and also to the Lebesgue integral.

Theorem 1. *A function f is RL integrable on $[a, b]$ if and only if there exists an absolutely continuous function F such that for every $\varepsilon > 0$ and $\eta > 0$ there exist an open set G and a constant $\delta > 0$ such that $|G| < \eta$ and that for every division $D = \{([u, v], \xi)\}$ with $0 < v - u < \delta$ and $\xi \in [u, v]$ we have*

$$(D) \sum_{\xi \notin G} |f(\xi)(v - u) - F(u, v)| < \varepsilon.$$

where $F(u, v)$ denotes $F(v) - F(u)$.

To prove Theorem 1, we need the following lemma.

Lemma 2. *A nonnegative function f is RL integrable on $[a, b]$ if and only if there exists a number A such that for every $\varepsilon > 0$ and $\eta > 0$ there exist an open set G and a division D_0 such that $|G| < \eta$ and that for any division $D = \{([u, v], \xi)\}$ finer than D_0 and $\xi \in [u, v]$ we have*

$$|(D) \sum_{\xi \notin G} f(\xi)(v - u) - A| < \varepsilon.$$

Proof. If f is nonnegative and RL integrable on $[a, b]$, choose $D_0 = \{[u, v]\}$ such that $0 < v - u < \delta$ and δ is given as in the definition of the RL integral. Then the necessity follows. For sufficiency, let f^N be the truncated function of f, i.e., $f^N(x) = f(x)$ when $f(x) \leq N$ and $f^N(x) = N$ when $f(x) > N$. Following the proof in [2; p.105] we can show that f^N is Henstock integrable on $[a, b]$. Then for every N there is A_N such that for every $\varepsilon > 0$ and $\eta > 0$ there exist an open set G_N and a division D_N such that $|G_N| < \eta$ and that for any division $D = \{([u,v],\xi)\}$ finer than D_N and D_0 with $\xi \in [u,v]$ we have

$$|(D) \sum_{\xi \notin G_N} f^N(\xi)(v-u) - A_N| < \varepsilon.$$

It follows that

$$\begin{aligned} A_N &< \varepsilon + (D) \sum_{\xi \notin G_N} f^N(\xi)(v-u) \\ &< \varepsilon + (D) \sum_{\xi \notin G} f^N(\xi)(v-u) + 2N\eta \\ &< \varepsilon + (D) \sum_{\xi \notin G} f(\xi)(v-u) + 2N\eta \\ &< 2\varepsilon + A + 2N\eta. \end{aligned}$$

Since ε and η are arbitrary, we obtain $A_N \leq A$ for all N. By the monotone convergence theorem for the Henstock integral, f is Henstock integrable and therefore RL integrable on $[a,b]$.

Proof of Theorem 1. The necessity follows from the fact that if f is RL integrable on $[a,b]$ then it is absolutely Henstock integrable there [2; p.105], and the required condition holds. To prove sufficiency, we first show that $|f|$ is RL integrable on $[a,b]$. Let $F^*(x)$ denote the total variation of F on $[a,x]$ for $x \in [a,b]$. Then for every $\varepsilon > 0$ there is a division D_0 of $[a,b]$ such that for any division D finer than D_0 we have

$$0 \leq F^*(a,b) - (D) \sum |F(u,v)| < \varepsilon.$$

We may choose $D_0 = \{[u,v]\}$ so that $0 < v - u < \delta$. Then for any $D = \{([u,v],\xi)\}$ finer than D_0 and $\xi \in [u,v]$ we have

$$\begin{aligned} &| \, (D) \sum_{\xi \notin G} |f(\xi)|(v-u) - F^*(a,b) \, | \\ &\leq | \, (D) \sum_{\xi \notin G} |f(\xi)|(v-u) - (D) \sum |F(u,v)| \, | \\ &\quad + | \, (D) \sum |F(u,v)| - F^*(a,b) \, | \\ &\leq (D) \sum_{\xi \notin G} |f(\xi)(v-u) - F(u,v)| + (D) \sum_{\xi \in G} |F(u,v)| \end{aligned}$$

$$+F^*(a,b) - (D)\sum |F(u,v)|$$
$$< \; 2\varepsilon + (D)\sum_{\xi \in G} |F(u,v)|.$$

The last term in the above inequality is small because F is absolutely continuous. Hence, by Lemma 2, $|f|$ is RL integrable on $[a,b]$.

Similarly, we can show that $|f| - f$ is RL integrable on $[a,b]$. Then so is f. The proof is complete.

Theorem 1 is interesting in two ways. It gives the Saks-Henstock lemma for the RL integral. Also, it gives a one-step definition for the RL integral without having to define non-negative functions first then the general case. Furthermore, as a consequence of Theorem 1, the integral of any RL integrable function, and therefore any Lebesgue integrable function, is the limit of a sequence of Riemann sums with **equally spaced** divisions and some suitable terms missing. More precisely, there is a sequence of partial divisions D_n, each of which is equally spaced, such that

$$\lim_{n\to\infty}(D_n)\sum f(\xi)(v-u) = \int_a^b f(x)dx.$$

Avoiding singularities is a standard technique in numerical integration. This is realized here by omitting some suitable terms.

To illustrate the use of the Saks-Henstock lemma (Theorem 1 above), we prove the following mean convergence theorem.

Theorem 3. *Let f_n be RL integrable on $[a,b]$ with $f_n(x) \to f(x)$ almost everywhere in $[a,b]$ as $n \to \infty$. If*

$$\int_a^b |f_n(x) - f_m(x)|dx \to 0 \text{ as } n,m \to \infty,$$

then f is RL integrable on $[a,b]$ and

$$\int_a^b f_n(x)dx \to \int_a^b f(x)dx \text{ as } n \to \infty.$$

Sketch of proof. Let $F_n(u,v)$ denote the RL integral of f_n on $[u,v] \subset [a,b]$. It is obvious that $\{F_n(u,v)\}$ is a Cauchy sequence, and the limit exists, denoted by $F(u,v)$. Given $\varepsilon > 0$, let

$$E_{m,n} = \{x; |f_m(x) - f_n(x)| \geq \varepsilon\}, \quad E_n = \{x; |f_n(x) - f(x)| \geq \varepsilon\}.$$

Then we have

$$\varepsilon |E_{m,n}| \leq \int_a^b |f_m(x) - f_n(x)| dx.$$

Taking $m \to \infty$ and then $n \to \infty$, we obtain $|E_n| \to 0$ as $n \to \infty$. Next, there exists n_0 such that $|F_n(a,b) - F(a,b)| < \varepsilon$ whenever $n \geq n_0$. Furthermore, there exists $\eta_0 > 0$ such that for any partial division D of $[a,b]$ with $(D)\sum |v - u| < \eta_0$ we have

$$\begin{aligned} (D)\sum |F_n(u,v)| &\leq \int_a^b |f_n(x) - f_{n_0}(x)| dx + (D)\sum |F_{n_0}(u,v)| \\ &< 2\varepsilon. \end{aligned}$$

Since f_n is RL integrable on $[a,b]$, by Theorem 1, for every $\eta > 0$ there exist G_n and $\delta_n > 0$ such that $|G_n| < \eta/2$ and that for any division $D = \{([u,v],\xi)\}$ with $0 < v - u < \delta_n$ and $\xi \in [u,v]$ we have

$$(D) \sum_{\xi \notin G_n} |f_n(\xi)(v-u) - F_n(u,v)| < \varepsilon.$$

We assume that $\eta \leq \eta_0$. Now choose a sufficiently large $N \geq n_0$ such that $|E_N| < \eta/2$. Then choose an open set $G \supset E_N \cup G_N$ such that $|G| < \eta$ and put $\delta = \delta_N$. Then for any division $D = \{([u,v],\xi)\}$ with $0 < v - u < \delta$ and $\xi \in [u,v]$ using the inequalities above we have

$$\begin{aligned}|(D)\sum_{\xi\notin G} f(\xi)(v-u)-F(a,b)| &\leq (D)\sum_{\xi\notin G}|f(\xi)-f_N(\xi)|(v-u)\\ &\quad +(D)\sum_{\xi\notin G}|f_N(\xi)(v-u)-F_N(u,v)|\\ &\quad +|F_N(a,b)-F(a,b)|+(D)\sum_{\xi\in G}|F_N(u,v)|\\ &< \varepsilon(b-a+4).\end{aligned}$$

That is, f is RL integrable to $F(a,b)$ on $[a,b]$.

By means of Theorem 3, other convergence theorems follow. Similarly, using the Saks-Henstock lemma we can prove that if f is RL integrable on $[a,b]$ then its primitive F is differentiable and equal to f almost everywhere [2; p.21]. In conclusion, we have demonstrated once again the power of the Saks-Henstock lemma in proving results of the Riemann-type integrals.

References

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