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RIEMANN TAILS AND THE LEBESGUE AND HENSTOCK INTEGRALS

The results in this paper are related to the concepts of globally small Riemann sums and functionally small Riemann sums as defined in [3]. We assume that the reader is familiar with the terminology of the Henstock integral. Let $f : [a, b] \to R$, let $E \subset [a, b]$, let δ be a positive function defined on $[a, b]$, and let $\mathcal{P} = \{(x_i, [c_i, d_i]) : 1 \le i \le q\}$ be a finite collection of non-overlapping tagged intervals in $[a, b]$. Then

$f(\mathcal{P}) = \sum_{i=1}^{q} f(x_i)(d_i - c_i)$ denotes the Riemann sum of f associated with $\mathcal{P}$;

χ_E denotes the characteristic function of E;

$\mathcal{C}E$ denotes the complement of E; and

$\mathcal{P}$ is E–subordinate to δ means that $\mathcal{P}$ is subordinate to δ and each of the tags x_i is in E.

We begin with a result that relates the Lebesgue and Henstock integrals. The proof is an adaptation of the proof of Lemma 2 that is found in [3]. It is an easy exercise to verify the following lemma.

LEMMA: Let $f : [a, b] \to R$, let A and B be measurable subsets of $[a, b]$ with $A \subset B$, and suppose that f is Lebesgue integrable on B. If L is a number between $\int_A f$ and $\int_B f$, then there exists a measurable set C such that $A \subset C \subset B$ and $\int_C f = L$.

THEOREM 1: If $f : [a, b] \to R$ is Henstock integrable on $[a, b]$, then for each $\epsilon > 0$ there exists a measurable set $E \subset [a, b]$ such that $\mu([a, b] - E) < \epsilon$, f is Lebesgue integrable on E, and $\int_E f = \int_a^b f$.

PROOF: To omit the trivial case, assume that f is not Lebesgue integrable on $[a, b]$. For each positive integer n, define sets

$$A_n = \{x \in [a, b] : n - 1 \le f(x) < n\}; \quad B_n = \{x \in [a, b] : -n \le f(x) < -n + 1\};$$

then let $a_n = \int_{A_n} f$ and $b_n = \int_{B_n} f$. Since $\bigcup_n (A_n \cup B_n) = [a, b]$, there exists an integer N such that the set $X = \bigcup_{n=1}^{N} (A_n \cup B_n)$ satisfies $\mu([a, b] - X) < \epsilon$. Since

f is bounded on X, it is Lebesgue integrable on X. Suppose that $\int_X f < \int_a^b f$; the case $\int_X f > \int_a^b f$ is similar. Since f is not Lebesgue integrable on $[a,b]$, the series $\sum_{n=1}^{\infty} a_n$ diverges. Choose an index $M > N$ such that

$$\int_X f + a_{N+1} + a_{N+2} + \cdots + a_M > \int_a^b f$$

and let $Y = X \cup A_{N+1} \cup \cdots \cup A_M$. Now f is Lebesgue integrable on Y and $\int_X f < \int_a^b f < \int_Y f$. By the lemma, there exists a measurable set $X \subset E \subset Y$ such that $\int_E f = \int_a^b f$. This completes the proof.

DEFINITION: A measurable function $f : [a,b] \to R$ is almost Lebesgue integrable on $[a,b]$ if for each $\epsilon > 0$ there exist a measurable set $E \subset [a,b]$ and a positive function δ on $[a,b]$ such that $\mu([a,b] - E) < \epsilon$, f is Lebesgue integrable on E, and $|f\chi_{cE}(\mathcal{P})| < \epsilon$ whenever $\mathcal{P}$ is a tagged partition of $[a,b]$ that is subordinate to δ.

THEOREM 2: A function $f : [a,b] \to R$ is Henstock integrable on $[a,b]$ if and only if it is almost Lebesgue integrable on $[a,b]$.

PROOF: Suppose first that f is Henstock integrable on $[a,b]$ and let $\epsilon > 0$. By Theorem 1, there exists a measurable set $E \subset [a,b]$ such that $\mu([a,b] - E) < \epsilon$, f is Lebesgue integrable on E, and $\int_E f = \int_a^b f$. Since f and $f\chi_E$ are Henstock integrable on $[a,b]$, there exists a positive function δ on $[a,b]$ such that

$$\left|f(\mathcal{P}) - \int_a^b f\right| < \epsilon/2 \quad \text{and} \quad \left|f\chi_E(\mathcal{P}) - \int_E f\right| < \epsilon/2$$

whenever $\mathcal{P}$ is a tagged partition of $[a,b]$ that is subordinate to δ. Let $\mathcal{P}$ be a tagged partition of $[a,b]$ that is subordinate to δ and compute

$$|f\chi_{cE}(\mathcal{P})| = |f(\mathcal{P}) - f\chi_E(\mathcal{P})| \le |f(\mathcal{P}) - \int_a^b f| + |\int_E f - f\chi_E(\mathcal{P})| < \epsilon.$$

Hence f is almost Lebesgue integrable on $[a,b]$.

Now suppose that f is almost Lebesgue integrable on $[a,b]$ and let $\epsilon > 0$. Choose a measurable set $E \subset [a,b]$ and a positive function δ_1 on $[a,b]$ such that $\mu([a,b] - E) < \epsilon$, f is Lebesgue integrable on E, and $|f\chi_{cE}(\mathcal{P})| < \epsilon/3$ whenever $\mathcal{P}$ is a tagged partition of $[a,b]$ that is subordinate to δ_1. Since $f\chi_E$ is Henstock integrable on $[a,b]$, there exists a positive function $\delta < \delta_1$ on $[a,b]$ such that $|f\chi_E(\mathcal{P}_1) - f\chi_E(\mathcal{P}_2)| < \epsilon/3$ whenever $\mathcal{P}_1$ and $\mathcal{P}_2$ are tagged partitions of $[a,b]$ that are subordinate to δ. For such partitions, we have

$$\begin{aligned} |f(\mathcal{P}_1) - f(\mathcal{P}_2)| &\le |f\chi_{cE}(\mathcal{P}_1)| + |f\chi_E(\mathcal{P}_1) - f\chi_E(\mathcal{P}_2)| + |f\chi_{cE}(\mathcal{P}_2)| \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon. \end{aligned}$$

By the Cauchy criterion for Henstock integrability, the function f is Henstock integrable on $[a, b]$.

Theorem 3 of [3] yields the following corollary:

COROLLARY: A measurable function $f : [a, b] \to R$ has functionally small Riemann sums if and only if it is almost Lebesgue integrable on $[a, b]$.

We next present two types of uniform integrability and two well-known convergence theorems.

DEFINITION: Let $\{f_n\}$ be a sequence of Lebesgue integrable functions defined on $[a, b]$. The sequence $\{f_n\}$ is uniformly Lebesgue integrable on $[a, b]$ if for each $\epsilon > 0$ there exists a positive integer N such that $\int_{\{|f_n|>N\}} |f_n| < \epsilon$ for all n.

Vitali Convergence Theorem: Let $\{f_n\}$ be a sequence of Lebesgue integrable functions defined on $[a, b]$ and suppose that $\{f_n\}$ converges pointwise to f on $[a, b]$. If the sequence $\{f_n\}$ is uniformly Lebesgue integrable on $[a, b]$, then f is Lebesgue integrable on $[a, b]$ and $\int_a^b f = \lim_{n\to\infty} \int_a^b f_n$.

DEFINITION: Let $\{f_n\}$ be a sequence of Henstock integrable functions defined on $[a, b]$. The sequence $\{f_n\}$ is uniformly Henstock integrable on $[a, b]$ if for each $\epsilon > 0$ there exists a positive function δ on $[a, b]$ such that $|f_n(\mathcal{P}) - \int_a^b f_n| < \epsilon$ for all n whenever $\mathcal{P}$ is a tagged partition of $[a, b]$ that is subordinate to δ.

Simple Convergence Theorem: Let $\{f_n\}$ be any sequence of Henstock integrable functions defined on $[a, b]$ and suppose that $\{f_n\}$ converges pointwise to f on $[a, b]$. If the sequence $\{f_n\}$ is uniformly Henstock integrable on $[a, b]$, then f is Henstock integrable on $[a, b]$ and $\int_a^b f = \lim_{n\to\infty} \int_a^b f_n$.

Several comments are in order at this point. The sequence $\{f_n\}$ is uniformly Lebesgue integrable on $[a, b]$ if and only if the sequence $\{\int_\cdot f_n\}$ is equi-absolutely continuous with respect to Lebesgue measure on $[a, b]$. The proof of this is a routine exercise. The statement of the Vitali convergence theorem given here is only a special case – the theorem actually is much stronger. See page 152 of [4]. The word simple in the convergence theorem for the Henstock integral refers to the fact that the proof is very easy. The theorem itself is quite powerful. See [1].

Now suppose that $f : [a, b] \to R$ is a measurable function. For each positive integer n, let

$$f_n(x) = \begin{cases} f(x), & \text{if } |f(x)| \le n; \\ 0, & \text{if } |f(x)| > n; \end{cases} \quad \text{and} \quad E_n = \{x \in [a, b] : |f(x)| > n\}.$$

Then $\{f_n\}$ converges pointwise to f, $f - f_n = f\chi_{E_n}$, and $\{E_n\}$ is a nonincreasing sequence of sets that converges to the empty set. If f is Lebesgue integrable on $[a,b]$, then the sequence $\{f_n\}$ is uniformly Lebesgue integrable on $[a,b]$ and $\int_a^b f = \lim_{n\to\infty} \int_a^b f_n$. However, if f is Henstock integrable on $[a,b]$, the sequence $\{f_n\}$ is not necessarily uniformly Henstock integrable on $[a,b]$ and $\int_a^b f$ may not equal $\lim_{n\to\infty} \int_a^b f_n$. An example of this phenomenon is given on page 115 of [2]. The function $f : [0,1] \to R$ is defined by

$$f(x) = \begin{cases} n^2, & \text{if } x \in (\frac{1}{n+1}, \frac{1}{n+1} + \frac{1}{n(n+1)^2}); \\ -n, & \text{if } x \in (\frac{1}{n+1} + \frac{1}{n(n+1)^2}, \frac{1}{n}); \\ 0, & \text{otherwise.} \end{cases}$$

See [2] for details. We look for conditions on f to guarantee that the sequence $\{f_n\}$ is uniformly Henstock integrable on $[a,b]$ and $\int_a^b f = \lim_{n\to\infty} \int_a^b f_n$.

DEFINITION: Let $f : [a,b] \to R$ be measurable and let $\{E_n\}$ be defined as above.

(a) The function f has small Riemann tails if for each $\epsilon > 0$ there exist a positive integer N and a positive function δ on $[a,b]$ such that $|f\chi_{E_n}(\mathcal{P})| < \epsilon$ for all $n \geq N$ whenever $\mathcal{P}$ is a tagged partition of $[a,b]$ that is subordinate to δ.

(b) The function f has really small Riemann tails if for each $\epsilon > 0$ there exist a positive integer N and a positive function δ on $[a,b]$ such that $|f(\mathcal{P})| < \epsilon$ whenever $\mathcal{P}$ is E_N-subordinate to δ.

As defined above, the concept of small Riemann tails is a uniform type of globally small Riemann sums. The difference between small Riemann tails and really small Riemann tails is subtle. Any $\mathcal{P}$ that is E_N-subordinate to δ can be extended to a tagged partition $\mathcal{P}_1$ of $[a,b]$ that is subordinate to δ. But since the extended partition may have more tags in E_N, it does not follow that $|f(\mathcal{P})| = |f\chi_{E_N}(\mathcal{P}_1)|$. This definition leads to the following results.

THEOREM 3: Let $f : [a,b] \to R$ be measurable and let $\{f_n\}$ be the sequence defined above. Then $\{f_n\}$ is uniformly Lebesgue integrable on $[a,b]$ if and only if f has really small Riemann tails.

PROOF: Suppose first that the sequence $\{f_n\}$ is uniformly Lebesgue integrable on $[a,b]$. It is easy to verify that $|f|_n = |f_n|$ and that the sequence $\{|f_n|\}$ is uniformly Lebesgue integrable on $[a,b]$. By the Vitali Convergence Theorem, the function $|f|$ is Lebesgue integrable on $[a,b]$ and $\lim_{n\to\infty} \int_a^b |f_n| = \int_a^b |f|$. It follows that

$$\lim_{n\to\infty} \int_a^b |f|\chi_{E_n} = \lim_{n\to\infty} \int_a^b (|f| - |f_n|) = 0.$$

Thus there exist a positive integer N and a positive function δ on $[a,b]$ such that $|f|\chi_{E_N}(\mathcal{P}_1) < \epsilon$ whenever $\mathcal{P}_1$ is a tagged partition of $[a,b]$ that is subordinate to δ. Suppose that $\mathcal{P}$ is E_N–subordinate to δ. Extend $\mathcal{P}$ to a tagged partition $\mathcal{P}_1$ of $[a,b]$ that is subordinate to δ. We then have

$$|f(\mathcal{P})| \leq |f|(\mathcal{P}) = |f|\chi_{E_N}(\mathcal{P}) \leq |f|\chi_{E_N}(\mathcal{P}_1) < \epsilon.$$

Hence f has really small Riemann tails.

Now suppose that f has really small Riemann tails. To prove that $\{f_n\}$ is uniformly Lebesgue integrable on $[a,b]$, it is sufficient to prove that f is Lebesgue integrable on $[a,b]$. This is equivalent to $|f|$ being Henstock integrable on $[a,b]$. We will prove that $|f|$ is almost Lebesgue integrable on $[a,b]$ and apply Theorem 2. Let $\epsilon > 0$. There exist a positive integer N_1 and a positive function δ on $[a,b]$ such that $|f(\mathcal{P})| < \epsilon/2$ whenever $\mathcal{P}$ is E_{N_1}–subordinate to δ. Choose an integer $N > N_1$ such that $\mu(E_N) < \epsilon$. Since f is bounded on $[a,b] - E_N$, the function $|f|$ is Lebesgue integrable on $[a,b] - E_N$. Let $\mathcal{P}$ be a tagged partition of $[a,b]$ that is subordinate to δ. Let $\mathcal{P}_N^+$ be the subset of $\mathcal{P}$ with the property: if x is a tag of $\mathcal{P}_N^+$, then $x \in E_N$ and $f(x) \geq 0$; and let $\mathcal{P}_N^-$ be the subset of $\mathcal{P}$ with the property: if x is a tag of $\mathcal{P}_N^-$, then $x \in E_N$ and $f(x) < 0$. Since both $\mathcal{P}_N^+$ and $\mathcal{P}_N^-$ are E_{N_1}–subordinate to δ,

$$|f|\chi_{E_N}(\mathcal{P}) = f(\mathcal{P}_N^+) - f(\mathcal{P}_N^-) < \epsilon/2 + \epsilon/2 = \epsilon.$$

Hence $|f|$ is almost Lebesgue integrable on $[a,b]$. This completes the proof.

THEOREM 4: Let $f : [a,b] \to R$ be measurable and let $\{f_n\}$ be the sequence defined above. Then $\{f_n\}$ is uniformly Henstock integrable on $[a,b]$ if and only if f has small Riemann tails.

PROOF: Suppose first that the sequence $\{f_n\}$ is uniformly Henstock integrable on $[a,b]$. By the Simple Convergence Theorem, the function f is Henstock integrable on $[a,b]$ and $\lim_{n\to\infty} \int_a^b f_n = \int_a^b f$. Let $\epsilon > 0$. Choose a positive function δ on $[a,b]$ such that

$$\left|f(\mathcal{P}) - \int_a^b f\right| < \epsilon/3 \quad \text{and} \quad \left|f_n(\mathcal{P}) - \int_a^b f_n\right| < \epsilon/3 \quad \text{for all } n$$

whenever $\mathcal{P}$ is a tagged partition of $[a,b]$ that is subordinate to δ and choose a positive integer N such that

$$\left|\int_a^b f_n - \int_a^b f\right| < \epsilon/3 \quad \text{for all } n \geq N.$$

Let $\mathcal{P}$ be a tagged partition of $[a,b]$ that is subordinate to δ and for $n \geq N$ compute

$$\begin{aligned} |f\chi_{E_n}(\mathcal{P})| &= |f(\mathcal{P}) - f_n(\mathcal{P})| \\ &\leq \left|f(\mathcal{P}) - \int_a^b f\right| + \left|\int_a^b f - \int_a^b f_n\right| + \left|\int_a^b f_n - f_n(\mathcal{P})\right| \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon. \end{aligned}$$

Hence f has small Riemann tails.

Now suppose that f has small Riemann tails and let $\epsilon > 0$. From the inequality,

$$\begin{aligned}\left|\int_a^b f_n - \int_a^b f_m\right| &\leq \left|\int_a^b f_n - f_n(\mathcal{P})\right| + \left|f_n(\mathcal{P}) - f(\mathcal{P})\right| + \left|f(\mathcal{P}) - f_m(\mathcal{P})\right| \\ &\quad + \left|f_m(\mathcal{P}) - \int_a^b f_m\right| \\ &= \left|\int_a^b f_n - f_n(\mathcal{P})\right| + \left|f\chi_{E_n}(\mathcal{P})\right| + \left|f\chi_{E_m}(\mathcal{P})\right| \\ &\quad + \left|f_m(\mathcal{P}) - \int_a^b f_m\right|,\end{aligned}$$

it is easy to verify that the sequence $\{\int_a^b f_n\}$ is Cauchy. Let $\epsilon > 0$. There exist a positive integer N_1 and a positive function δ_1 on $[a, b]$ such that $|f\chi_{E_n}(\mathcal{P})| < \epsilon/4$ for all $n \geq N_1$ whenever $\mathcal{P}$ is a tagged partition of $[a, b]$ that is subordinate to δ_1. Choose an integer $N > N_1$ such that

$$\left|\int_a^b f_n - \int_a^b f_m\right| < \epsilon/4 \quad \text{for all } m, n \geq N$$

and choose a positive function $\delta < \delta_1$ on $[a, b]$ such that

$$\left|f_n(\mathcal{P}) - \int_a^b f_n\right| < \epsilon \quad \text{for } 1 \leq n \leq N$$

whenever $\mathcal{P}$ is a tagged partition of $[a, b]$ that is subordinate to δ. Suppose that $\mathcal{P}$ is a tagged partition of $[a, b]$ that is subordinate to δ and for $n > N$ compute

$$\begin{aligned}\left|f_n(\mathcal{P}) - \int_a^b f_n\right| &\leq \left|f_n(\mathcal{P}) - f(\mathcal{P})\right| + \left|f(\mathcal{P}) - f_N(\mathcal{P})\right| + \left|f_N(\mathcal{P}) - \int_a^b f_N\right| \\ &\quad + \left|\int_a^b f_N - \int_a^b f_n\right| \\ &= \left|f\chi_{E_n}(\mathcal{P})\right| + \left|f\chi_{E_N}(\mathcal{P})\right| + \left|f_N(\mathcal{P}) - \int_a^b f_N\right| \\ &\quad + \left|\int_a^b f_N - \int_a^b f_n\right| \\ &< \epsilon/4 + \epsilon/4 + \epsilon/4 + \epsilon/4 = \epsilon.\end{aligned}$$

Hence $\{f_n\}$ is uniformly Henstock integrable on $[a, b]$.

The following theorem is proved in [2].

THEOREM 5: Let $f : [a, b] \to R$ be measurable and let $\{f_n\}$ be the sequence defined above. Then f is Henstock integrable on $[a, b]$ and $\int_a^b f = \lim_{n\to\infty} \int_a^b f_n$ if and only if f has globally small Riemann sums.

We conclude this paper with an open question. The function given earlier is an example of a function that is Henstock integrable but does not have globally small Riemann sums. Is there a function that has globally small Riemann sums but does not have small Riemann tails? That is, is it possible for f to be Henstock integrable on $[a, b]$ and $\int_a^b f = \lim_{n\to\infty} \int_a^b f_n$, but that the sequence $\{f_n\}$ is not uniformly Henstock integrable on $[a, b]$?

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