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THE PACKING MEASURE AND FUBINI'S THEOREM

The question arose as to whether Fubini's Theorem holds in the plane with respect to the packing measure. In other words, if G is a closed set in the unit square, then is the packing measure of G equal to the integral of the packing measure of $G[x] = \{y : (x, y) \in G\}$ with respect to the packing measure of the projection of G onto the x-axis. The answer is no. The integral may be an equivalent measure to the packing measure (the two measures are at the same time zero, finite, of infinite on a set G) only when the dimension of the cartesian product of sets is equal to the sum of the dimensions of the original sets (e.g., when Dim $(E \times F) =$ Dim $E +$ Dim F). To prove this assertion, the following definitions will be needed.

Definition 1: Packing Measure. The premeasure is $(\alpha - \underline{P})(E) = \inf_{\delta \to 0}$ $\{\sup[\sum_i (2r_i)^\alpha : B(x_i, r_i) = (x_i - r_i, x_i + r_i)$ is any disjoint sequence of open intervals, or open circles in the plane, with $x_i \in E$ and $r_i < \delta]\}$. The packing measure is $(\alpha - p)(E) = \inf\{\sum_i (\alpha - \underline{P})(E_i) : E \subset \cup_i E_i\}$.

Definition 2: Symmetric derivation basis measure. Let $\delta(x)$ be a positive real function. Then, $(\alpha - \underline{S})_\delta(E) = \sup\{\sum_{i=1}^n (2r_i)^\alpha : B(x_i, r_i) = (x_i - r_i, x_i + r_i)$ is any disjoint sequence of open intervals, or open circles in the plane, with $x_i \in E$ and $r_i < \delta(x_i)\}$. Then, the symmetric derivation basis measure is $(\alpha - s)(E) = \inf\{(\alpha - \underline{S})_\delta(E) : \delta(x)$ is any positive real function$\}$.

It was shown in [1] that the packing measure is the symmetric derivation basis measure.

Definition 3: The Integral measure. Let $\delta(x, y) > 0$ be a positive real function defined on the plane. Then, for $\gamma = \alpha + \beta$, $(\gamma - \hat{\underline{S}}_\delta(G) = \sup\{\sum_{i,j} (2r_i)^\alpha (2t_j)^\beta$: the supremum is over all disjoint sequences $R(x_i, y_j; r_i, t_j) = (x_i - r_i, x_i + r_i) \times (y_j - t_j, y_j + t_j)$ of open rectangles with center $(x_i, y_j) \in G$ and with sides of length $2r_i, 2t_j$ such that $R(x_i, y_j; r_i, t_j) \subset B(x_i, y_j; \delta(x_i, y_j))$, where $B(x_i, y_j; \delta(x_i, y_j))$ is a circle with center (x_i, y_j) and radius $\delta(x_i, y_j)\}$. Then the integral measure is

$(\gamma - \hat{s})(G) = \inf\{(\gamma - \hat{\underline{S}})_\delta(G) : \delta(x,y)$ is any positive, real function defined on the plane$\}$.

As suggested by the name given to the outer measure defined in Definition 3, the integral measure is the integral discussed above.

First, let E and F be subsets of the interval $[0,1]$. The following theorem shows that $\int(\alpha - p)(E)d(\beta - p)(F) = (\alpha - p)(E) \cdot (\beta - p)(F)$ is the integral measure. It should be stated that in general, it is false that $[(\alpha+\beta) - p](E \times F) = (\alpha - p)(E) \cdot (\beta - p)(F)$.

Theorem 1. *Let E and F be sets in the interval* $[0,1]$. *If $(\alpha - p)(E)$ and $(\beta - p)(F)$ are finite and non-zero, then $(\gamma - \hat{s})(E \times F) = (\alpha - p)(E) \cdot (\beta - p)(F)$ where $\gamma = \alpha + \beta$.*

Proof. In the following proof, $(\alpha - s)(E)$ and $(\beta - s)(F)$ will be used since the symmetric derivation basis measure is equal to the packing measure. Let $\varepsilon > 0$ be given and choose $\delta_1(x)$ such that $(\alpha - \underline{S})_{\delta_1}(E) < (\alpha - s)(E) + \varepsilon$ and $(\beta - \underline{S})_{\delta_1}(F) < (\beta - s)(F) + \varepsilon$. Then, there exists a packing $\{B(x_i, r_i) = (x_i - r_i, x_i + r_i)$ with $x_i \in E$ and $r_i < \delta_1(x_i)\}$ such that $(\alpha - s)(E) - \varepsilon < \sum_i (2r_i)^\alpha$. Also, there exists a packing $\{B(y_i, t_j) = (y_j - t_j, y_j + t_j)$ with $y_j \in F$ and $t_j < \delta_1(y_j)\}$ such that $(\beta - s)(F) - \varepsilon < \sum_j (2t_j)^\beta$. Therefore, $[(\alpha - s)(E) - \varepsilon][(\beta - s)(F) - \varepsilon] < \sum_{i,j} (2r_i)^\alpha (2t_j)^\beta \leq (\gamma - \hat{\underline{S}})_{\delta_0}(E \times F)$ where $\delta_0(x_i, y_j) = \max\{\sqrt{2}\,\delta_1(x_i),\ \sqrt{2}\,\delta_1(y_j)\}$. Since the above can be done with any $\delta'(x) < \delta_1(x)$ and $\delta''(y) < \delta_1(y)$, $(\alpha - s)(E)(\beta - s)(F) \leq (\gamma - \hat{s})(E \times F)$. Now, let $\delta(x,y)$ be any positive real bounded function. Then, there exists a packing of rectangles $R(x_i, y_j; r_i, t_j) \subset B(x_i, y_j; \delta(x_i, y_j))$ such that $(\gamma - \hat{s})(E \times F) - \varepsilon < \sum_{i,j} r_i^\alpha t_j^\beta \leq (\alpha - \underline{S})_{\delta_2}(E) \cdot (\beta - \underline{S})_{\delta_3}(F)$ where $\delta_2(x) \geq \{\delta(x,y) : y \in F\}$ and $\delta_3(y) \geq \{\delta(x,y) : x \in E\}$. Since $\delta(x,y)$ was arbitrary, $(\gamma - \hat{s})(E \times F) \leq (\alpha - s)(E) \cdot (\beta - s)(F)$. Thus, $(\gamma - \hat{s})(E \times F) = (\alpha - s)(E) \cdot (\beta - s)(F)$.

One referee asks the question as to whether Theorem 1 can be extended to the case where E or F has measure 0.

Notice that $(\gamma - \hat{s})(E \times F)$ is greater than or equal to a packing using squares with centers in $E \times F$. A packing with squares centered on $E \times F$ (calculated as in Definition 3) is equivalent to the packing measure only when the packing measure dimensions add (e.g. $\mathrm{Dim}(E \times F) = \mathrm{Dim}\, E + \mathrm{Dim}\, F$).

Now for the theorem that shows for a set G in the unit square, the integral of the packing measures of $G[x]$ with respect to the packing measure of the projection of G onto the x-axis is the integral measure.

Theorem 2. *Let G be a closed set. Let E be the projection of G onto the y-axis and let F be the projection of G onto the x-axis. Set the Dim $F = \beta$ and assume that $\alpha =$ Dim $E =$ Dim $G[x]$ for each $x \in F$. If $\gamma = \alpha + \beta$ and $(\gamma - \hat{s})(E \times F) < \infty$, then $\int(\alpha - p)(G[x])d(\beta - p)(F) = (\gamma - \hat{s})(G)$.*

Proof. In Theorem 1, it was shown that $\int(\alpha - p)(E)d(\beta - p)(F) = (\gamma - \hat{s})(E \times F)$. Let H_n be the collection of all dyadic squares with sides equal to 2^{-n} and contained in the complement of G. If $G = (E \times F) \sim Q$, where Q is a dyadic square, then Theorem 2 is true since G can be divided into at most four disjoint rectangles and Theorem 1 can thus be applied. Let now $G_n = (E \times F) \sim H_n$. Therefore, an induction argument shows that

$$(\gamma - \hat{s})(G_1) = \int(\alpha - p)(G_1[x])d(\beta - p)(F)$$

and likewise,

$$(\gamma - \hat{s})(G_n) = \int(\alpha - p)(G_n[x])d(\beta - p)(F).$$

Since $(\gamma - \hat{s})(E \times F) < \infty$ and $\{G_n\}$ decreases montonically to G,

$$\lim_{n\to\infty}(\gamma - \hat{s})(G_n) = (\gamma - \hat{s})(G)$$

and

$$\lim_{n\to\infty}\int(\alpha - p)(G_n[x])d(\beta - p)(F) = \int(\alpha - p)(G[x])d(\beta - p)(F).$$

Thus, $(\gamma - \hat{s})(G) = \int(\alpha - p)(G[x])d(\beta - p)(F)$.

In Theorem 2, one referee asks the question as to whether it is true that the set F will always have the property $0 < (\beta - p)(F) < +\infty$.

The question of what measure is the integral $\int(\alpha_x - p)(G[x])d(\beta - p)(F)$ when $\alpha_x =$ Dim $G[x]$ varies with each x was not considered by the author. It is conjectured that it is the integral measure which would be equivalent to the packing measure under very restrictive conditions.

It is known that if E is the Cantor Set, then Dim $(E \times E) =$ Dim $E +$ Dim E. So, the integral measure is equivalent to the packing measure of $E \times E$ in the plane. An example of sets E and F such that the integral measure of $E \times F$ is not equivalent to the packing measure of $E \times F$ in the plane is the subject of the following example.

Example. There exists a set E with packing measure dimension, Dim $E = \log 2/2\log(2s)$, and a set F with Dim $F = \log 2/3\log(2s)$, $s > 1$. The packing measure dimension of $E \times F$ is Dim $(E \times F) \leq 2\log 2/3\log(2s) <$ Dim $E +$ Dim F.

Proof. Construct the Cantor-like set E as follows: Remove an open interval from the center of the interval $[0,1]$ leaving two equal closed intervals of length $a_1 = (2s)^{-5}$. Remove an open interval from the center of these two closed intervals leaving 4 closed intervals of length $a_2 = (2s)^{-6}$. Continue this method so that $a_3 = (2s)^{-7}$ and $a_4 = (2s)^{-8}$. Notice that $-8 = -2 \cdot 4 = -2 \cdot n$. Form $a_5 = (2s)^{-25}$ where $-25 = -5 \cdot 5 = -5n$. Then, continue as before: $a_6 = (2s)^{-26}$, $a_7 = (2s)^{-27}, \ldots, a_{20} = (2s)^{-40}$, and then $a_{21} = (2s)^{-105}$. Therefore, for each n, $(2s)^{-5n} \le a_n \le (2s)^{-2n}$ and Dim $E = \log 2/2\log(2s)$, [3]. For the set F, do a similar construction to E where $a_1 = (2s)^{-5}$, $a_2 = (2s)^{-6}$, $a_3 = (2s)^{-15}$, $a_4 = (2s)^{-16}$, $a_5 = (2s)^{-17}$, $a_6 = (2s)^{-18}$, $a_7 = (2s)^{-35}$, $a_8 = (2s)^{-36}$, $a_9 = (2s)^{-37}, \ldots, a_{14} = (2s)^{-42}$, $a_{15} = (2s)^{-75}$, and continue with $a_{16} = (2s)^{-76}$. Therefore, $(2s)^{-5n} \le a_n \le (2s)^{-3n}$, for each n, and hence Dim $F = \log 2/3 \cdot \log(2s)$. Examine a grid formed with the closed intervals of length a_n^E and a_n^F. The maximum size of the rectangles is $(2s)^{-3n} \times (2s)^{-2n}$. Using squares with side $(2s)^{-3n}$, the packing measure dimension of $E \times F$ [3] would be

$$\frac{\log 2(2^n)^2}{-\log \sqrt{2}(2s)^{-3n}} = \frac{(2n+1)\cdot \log 2}{3n \cdot \log(2s) - \log \sqrt{2}} \longrightarrow \frac{2 \cdot \log 2}{3 \cdot \log(2s)}$$

as $n \to \infty$. The most irregular rectangle is of size $(2s)^{-5n} \times (2s)^{-2n}$ and using squares of sides $(2s)^{-5n}$, then $\log 3(2^n)^2$ would be used for the numerator of the above fraction. So, the $\mathrm{Dim}(E \times F) \le (2 \cdot \log 2)/(3 \cdot \log(2s))$ and Dim $E +$ Dim $F = (5 \cdot \log 2)/(6 \cdot \log(2s))$.

References

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[3] C. Tricot, *Two definitions of fractional dimension*, Math. Proc. Camb. Phil. Soc., 91 (1982), 52-74.

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