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Marek Balcerzak, Institute of Mathematics, Łódź University, ul. Stefana Banacha 22, 90-238 Łódź, Poland.

ANOTHER NONMEASURABLE SET WITH PROPERTY (s^0)

__Abstract__. It is observed that the Walsh set $E \subseteq X \times Y$ which has property (s^0) and is nonmeasurable can satisfy the following conditions simultaneously: E is nonmeasurable in some strong sense; E is the graph of a bijection; E meets each function of a fixed family of size c, in a set of size $< c$.

In the paper, X and Y are uncountable Polish (i.e. metric separable and complete) spaces. A set $A \subseteq X$ is said to have property (s^0) (or is called an (s^0) set) if any perfect set $P \subseteq X$ contains a perfect set Q disjoint from A (see [Sz] and also [Mi], [W1]). All (s^0) sets in X form a σ-ideal. In [W2] Walsh gave an example of an (s^0) set $E \subseteq X \times Y$ that has not the Baire property and is not $(\mu \times \nu)$-measurable for any continuous finite Borel measures μ and ν on X and Y (respectively). The same result was obtained independently by Corazza in [Co]. It is well known that there exists a nonmeasurable set in the plane, $\mathbf{R}^2$, being the graph of a bijection, i.e. meeting each of the horizontal and vertical lines at at most one point (this is due to Sierpiński who, in fact, showed a stronger version with the lines of arbitrary slope, cf. [S]). Recently, a simple proof of that result has been given in [vD]. We mix techniques of [W2] and [vD] to get the above effects and some additional properties. Namely, our set does not belong to a wide class of σ-algebras associated with Fubini products of σ-ideals (here Lemma 2 derived from [B3] is helpful). Moreover, we can ensure that, for a fixed family $\mathcal{F}$ of functions $f : X \to Y$ such that $|\mathcal{F}| = c$ (the cardinality of the continuum), our set meets each $f \in \mathcal{F}$ in a set of size $< c$ (we identify f with its graph). If one considers the σ-ideal $\mathcal{T}$ generated by $\mathcal{F}$ (i.e. consisting of all sets A such that $A \subseteq \bigcup\mathcal{F}^*$ for a countable $\mathcal{F}^* \subseteq \mathcal{F}$), then our set E meets each member of $\mathcal{T}$ in a set of size $< c$ (hence E forms a c-$\mathcal{T}$-Lusin set in the notation proposed in [Mi]). That property of E is established similarly as in [GL] and has the related application: it yields a nonmeasurable function $F : \mathbf{R}^2 \to \mathbf{R}$ with "nice" superpositions $F(x, f(x))$ for all Borel $f : \mathbf{R} \to \mathbf{R}$ (cf. condition (iii) in Example $\mathbf{4^0}$ below; see also [B3]).

By $\mathcal{B}(X)$ we denote the family of all Borel sets in X. If $\mathcal{T}$ is a σ-ideal of subsets of X, then $S(\mathcal{T})$ denotes the smallest σ-algebra containing $\mathcal{B}(X)$ and $\mathcal{T}$; $S(\mathcal{T})$ consists of all sets of the form $(B \setminus A) \cup (A \setminus B)$ where $B \in \mathcal{B}(X)$ and $A \in \mathcal{T}$.

For σ-ideals $\mathcal{T}$ and $\mathcal{J}$ of subsets of X and Y, respectively, we define (cf. [CKP])

$$\mathcal{T} \otimes \mathcal{J} = \{A \subseteq X \times Y : \exists\, B \in \mathcal{B}(X \times Y)(A \subseteq B \ \&\ \{x \in X : B_x \notin \mathcal{J}\} \in \mathcal{T})\}$$

where $B_x = \{y \in Y : \langle x, y\rangle \in B\}$ for $x \in X$. The family $\mathcal{T} \otimes \mathcal{J}$ forms a σ-ideal called the Fubini product of $\mathcal{T}$ and $\mathcal{J}$.

The σ-ideal of all countable subsets of X will be written as $\mathcal{T}_0^X$.

Theorem. *Let $\mathcal{F}$ be a set of functions from X to Y such that $|\mathcal{F}| = c$. There is an $E \subseteq X \times Y$ such that*

(1) *$|E| = c$ and E has property (s^0),*

(2) *E is the graph of a bijection $g : U \to W$ where $U \subseteq X$, $|U| = c$ and $W \subseteq Y$,*

(3) *$|E \cap f| < c$ for any $f \in \mathcal{F}$,*

(4) *$E \notin S(\mathcal{T} \otimes \mathcal{J})$ for any σ-ideals $\mathcal{T} \supseteq \mathcal{T}_0^X$ and $\mathcal{J} \supseteq \mathcal{T}_0^Y$.*

We need two lemmas.

Lemma 1. (cf. [W1], Th. 2.2). *If $\{D_\alpha : \alpha < c\}$ is a disjoint family of uncountable Borel sets in a Polish space and $\{E_\alpha : \alpha < c\}$ consists of all perfect sets P such that $|P \cap D_\alpha| < c$ for each $\alpha < c$, then*

(a) *$|D_\alpha \setminus \bigcup_{\beta<\alpha} E_\beta| = c$ for all $\alpha < c$,*

(b) *any selector of the family $\{D_\alpha \setminus \bigcup_{\beta<\alpha} E_\beta : \alpha < c\}$ has property (s^0).* ■

Lemma 2. ([B3], Lemma 1.6). *If E is the graph of a function $g : A \to Y$, $A \subseteq X$, which meets each member of $\mathcal{B}(X \times Y) \setminus (\mathcal{T}_0^X \otimes \mathcal{T}_0^Y)$, then $E \notin S(\mathcal{T} \otimes \mathcal{J})$ for any σ-ideals $\mathcal{T} \supseteq \mathcal{T}_0^X$ and $\mathcal{J} \supseteq \mathcal{T}_0^Y$.* ■

Proof of Theorem. Arrange all sets from $\mathcal{B}(X \times Y) \setminus (\mathcal{T}_0^X \otimes \mathcal{T}_0^Y)$ in $\{B_\alpha : \alpha < c\}$. For any $\alpha < c$, the set $A_\alpha = \{x \in X : (B_\alpha)_x \notin \mathcal{T}_0^Y\}$ is analytic (see [K], p. 496) and since it is uncountable, $|A_\alpha| = c$ (see [K], §39 I). Pick $x_0 \in A_0$ and $x_\alpha \in A_\alpha \setminus \{x_\beta : \beta < \alpha\}$ for $0 < \alpha < c$. Denote $D_\alpha = (\{x_\alpha\} \times Y) \cap B_\alpha$ for $\alpha < c$. Arrange all functions from $\mathcal{F}$ in $\{f_\alpha : \alpha < c\}$. Choose $p_0 = \langle x_0, y_0\rangle \in D_0$ and, for $0 < \alpha < c$, pick $p_\alpha = \langle x_\alpha, y_\alpha\rangle \in (D_\alpha \setminus \bigcup_{\beta<\alpha} E_\beta) \setminus \bigcup_{\beta<\alpha} \{\langle x_\alpha, f_\beta(x_\alpha)\rangle, \langle x_\alpha, y_\beta\rangle\}$ (where the E_α's are defined as in Lemma 1). That is possible by Lemma 1 (a). Put $E = \{p_\alpha : \alpha < c\}$.

Now, (1) follows from Lemma 1 (b). Assertion (2) is a consequence of the construction. To show (3), fix any $f_\beta \in \mathcal{F}$ and observe that $E \cap f_\beta \subseteq \bigcup_{\alpha<\beta} \{\langle x_\alpha, f_\beta(x_\alpha)\rangle\}$. Since E meets every B_α, by Lemma 2, we get (4). ■

Examples

1° Let $\mathcal{K}_X$ and $\mathcal{K}_Y$ denote the σ-ideals of all meager sets, i.e. those of the first category in X and Y, respectively. We can consider them as $\mathcal{T}$ and $\mathcal{J}$ in assertion (4) of the theorem. By the Kuratowski-Ulam theorem (see [0]), the product $\mathcal{K}_X \otimes \mathcal{K}_Y$ forms the σ-ideal of all meager sets in $X \times Y$. The family $S(\mathcal{K}_X \otimes \mathcal{K}_Y)$ consists of all sets with the Baire property in $X \times Y$.

2° Let μ and ν be σ-finite continuous Borel measures defined on X and Y, respectively, and let $\mathcal{T}_\mu$ and $\mathcal{T}_\nu$ denote the families of their null sets. Then, by the Fubini theorem, $\mathcal{T}_\mu \otimes \mathcal{T}_\nu$ is the family of all null sets with respect to $\mu \times \nu$, and $S(\mathcal{T}_\mu \otimes \mathcal{T}_\nu)$ forms the σ-algebra of all $(\mu \times \nu)$-measurable sets.

3° Let μ be the Lebesgue measure in $\mathbf{R}$. For brevity, write $\mathcal{T}_\mu = \mathcal{L}$ and $\mathcal{K}_{\mathbf{R}} = \mathcal{K}$ (cf. 1° and 2°). The four products $\mathcal{K}\otimes\mathcal{K}$, $\mathcal{L}\otimes\mathcal{L}$, $\mathcal{K}\otimes\mathcal{L}$, $\mathcal{L}\otimes\mathcal{K}$ are incomparable in the sense of inclusion (see [Me]) and the same can be observed for the respective σ-algebras $S(\mathcal{K}\otimes\mathcal{K})$, $S(\mathcal{L}\otimes\mathcal{L})$, $S(\mathcal{K}\otimes\mathcal{L})$, $S(\mathcal{L}\otimes\mathcal{K})$ (see [B1]). In [B2] we gave examples of sets $E_1 \notin S(\mathcal{K}\otimes\mathcal{L})$ and $E_2 \notin S(\mathcal{L}\otimes\mathcal{K})$ which are the graphs of bijections. The present construction leads to one set good for both the examples.

4° Let $X = Y = \mathbf{R}$ and let F be the characteristic function of E from the theorem. We then have:

(i) $\{\langle x,y\rangle \in \mathbf{R}^2 : F(x,y) \neq 0\}$ is an (s^0) set of size c,

(ii) all the functions

$$x \mapsto F(x, y_0), \qquad y \mapsto F(x_0, y),$$

for $x_0 \in X$, $y_0 \in Y$, are upper semicontinuous (since each of them equals zero everywhere except for at most one point),

(iii) for any Borel $f : \mathbf{R} \to \mathbf{R}$, the superposition $x \mapsto F(x, f(x))$ equals zero everywhere except for a set of size $< c$ (note that each set of size $< c$ has property (s^0), see [W1], Th. 2.1),

(iv) F is neither Lebesgue measurable nor has the Baire property.

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