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ON JOINT SUMMABILITY OF FOURIER SERIES AND CONJUGATE SERIES

1 Introduction.

It is well known that in spite of Fejers' theorem guaranteeing that a continuous function on $[-\pi, \pi]$ is uniformly approximable by its $(C, 1)$ means, these means are generally poor approximations in the sense of rate of convergence. The following theorem to be found on p. 122 of Zygmund [2] demonstrates this fact. In this note we use Zygmunds notations A_n^α, $K_n^\alpha(t)$, $\tilde{K}_n^\alpha(t)$, $D_n(t)$, $\tilde{D}_n(t)$, for the cesaro numbers, (C, α) kernel, conjugate (C, α) kernel, Dirichlet kernel and conjugate Dirichlet kernel respectively. The notations $\sigma_n^\alpha(f; x)$, $s_n(f; x)$ are used for the nth (C, α) mean and nth partial sum for the Fourier series for f at x.

Theorem 1. *Let $f \in L^1([-\pi, \pi])$ be periodic. If $\sigma_n^1(f; x) - f(x) = o(\frac{1}{n})$ uniformly as $n \to \infty$, then f is almost everywhere equal to its zeroth Fourier coefficient $c_0(f)$.*

In other words for general functions, the rate of convergence is slower than $o(\frac{1}{n})$. The following theorem is a pointwise convergence variant of this fact. It implies that it is difficult for a function of the kind considered to have both rapidly converging (C, α) means and conjugate means, even at a point.

Theorem 2. *Fix $x \in [-\pi, \pi]$. Let $f(x+t)\cot t/2 \in L^1([-\pi, \pi], dt)$. Assume that if $h(x)$ be any of the three functions $f(x)$, $f(x)\sin x$, $f(x)\cos x$ the following hold for some $\alpha > -1$*

$$\sigma_n^{\alpha+1}(h; x) - h(x) = o(\frac{1}{n})$$

and

$$\tilde{\sigma}_n^{\alpha+1}(h; x) - \tilde{h}(x) = o(\frac{1}{n}).$$

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Then

$$\sigma_n^\alpha(f;x) \to \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)dt$$

and

$$\tilde{\sigma}_n^\alpha(f;x) \to 0.$$

The proof of Theorem 2 will proceed from the results of the next section.

Remark 1. Since the hypotheses involve higher order summability than the conclusion, the theorem may be viewed as a Tauberian theorem with the hypothesized rate of convergence as the Taubarian condition.

Remark 2. The first hypothesis can be better appreciated in view of the fact that for almost all $x \in [-\pi,\pi]$ the integral $\int f(x+t)\cot(t/2)dt$ exists as a principal value integral. (See Zygmund [1], p. 131.)

2 Integral representation and a recurrence formula for the Cesaro kernel.

For fixed $t \in [-\pi,\pi]$, $\alpha > -1$ consider the function

$$h_t(z) = \frac{(1-z)^{-\alpha}}{1-e^{it}z}$$

on the unit disk. By analytic continuation we may write:

$$(1-z)^{-\alpha} = \sum_{n=0}^{\infty} A_n^{\alpha-1} z^n; \quad |z| < 1.$$

So, the nth Taylor coefficient of $h_t(z)$ at the origin is:

$$C_n\left[\frac{(1-z)^{-\alpha}}{1-e^{it}z}\right] = \sum_{k=0}^{n} A_k^{\alpha-1} e^{i(n-k)t}.$$

But from Cauchy theory:

$$C_n\left[\frac{(1-z)^{-\alpha}}{1-e^{it}z}\right] = \frac{1}{2\pi i}\int_{C(r)} \frac{(1-z)^{-\alpha}}{1-e^{it}z}\cdot\frac{1}{z^{n+1}}dz$$

where $\mathcal{C}(r)$ is a circle at 0 of radius $r < 1$. Hence,

$$\begin{aligned}
&\operatorname{Im}\left\{\frac{e^{it/2}}{2A_n^\alpha \sin(t/2)} \cdot \frac{1}{2\pi i}\int_{C(r)} \frac{(1-z)^\alpha}{1-e^{it}z}\,\frac{1}{z^{n+1}}dz\right\} \\
&= \operatorname{Im}\left\{\frac{1}{2A_n^\alpha \sin(t/2)} \sum_{k=0}^n A_k^{\alpha-1}\, e^{i(n-k+1/2)t}\right\} \\
&= \frac{1}{A_n^\alpha} \sum_{k=0}^n A_k^{\alpha-1}\, D_{n-k}(t) = K_n^\alpha(t).
\end{aligned}$$

Recalling that

$$\tilde{K}_n^\alpha(t) = \frac{\cot(t/2)}{2} - \frac{1}{A_n^\alpha}\sum_{k=0}^n A_{n-k}^{\alpha-1}\,\frac{\cos(k+\frac{1}{2})t}{2\sin(t/2)}$$

and performing a calculation similar to the above with Re replacing Im, it is seen that the following lemma holds.

Lemma 1. *Let $t \in [-\pi,\pi]$, $\alpha > -1$ and let $I_n^\alpha(t)$ denote the integral*

$$\frac{1}{2\pi i}\int_{C(r)} \frac{(1-z)^{-\alpha}}{1-e^{it}z}\,\frac{1}{z^{n+1}}\,dz.$$

Then

$$K_n^{-\alpha}(t) = \operatorname{Im}\left\{\frac{e^{it/2}}{2A_n^\alpha \sin(t/2)}\, I_n^\alpha(t)\right\} \tag{1}$$

$$\frac{\cot(t/2)}{2} - \tilde{K}_n^\alpha(t) = \operatorname{Re}\left\{\frac{e^{it/2}}{2A_n^\alpha \sin(t/2)}\, I_n^\alpha(t)\right\}.$$

Note: Formula (1) is equivalent to formula (2.3) of appendix II of Hardy [1]. This formula involves the Poisson kernel and makes the possiblitities for the recursion in Lemma 2 below less clear.

Next a recurrence formula in α will be found for $I_n^\alpha(t)$. Note that

$$\frac{1}{1-ze^{it}} - \frac{1}{1-z} = \frac{z(e^{it}-1)}{(1-ze^{it})(1-z)}$$

so

$$\int_{C(r)} \frac{(1-z)^{-\alpha}}{1-e^{it}z}\cdot\frac{1}{z^{n+1}}\,dz = \int_{C(r)} \frac{(1-z)^{-\alpha-1}}{z^{n+1}}dz + (e^{it}-1)\int_{C(r)} \frac{(1-z)^{-\alpha-1}}{(1-ze^{it})}\cdot\frac{1}{z^n}dz.$$

Recognizing the last integral as $2\pi i I_{n-1}^{\alpha+1}(t)$ and calculating:

$$\frac{1}{2\pi i}\int_{C(r)} \frac{(1-z)^{-\alpha-1}}{z^{n+1}}dz = (-1)^n\binom{-\alpha-1}{n} = \binom{n+\alpha}{n}$$

we have the following recurrence formula.

Lemma 2. *For* $t \in [-\pi, \pi]$, $\alpha > -1$;

$$I_n^\alpha(t) = \binom{n+\alpha}{n} + (e^{it} - 1) I_{n-1}^{\alpha+1}(t).$$

This recurrence leads to the following one for $K_n^\alpha(t)$ and $\tilde{K}_n^\alpha(t)$.

Lemma 3. *Let* $t \in [-\pi, \pi]$, $\alpha > -1$. *Then*

$$K_n^\alpha(t) = \frac{1}{2} + \frac{n}{\alpha+1}(\cos t - 1) K_{n-1}^{\alpha+1}(t) + \frac{n}{\alpha+1} \sin t [\frac{\cot(t/2)}{2} - \tilde{K}_{n-1}^{\alpha+1}(t)]$$

$$\tilde{K}_n^\alpha(t) = \frac{n}{\alpha+1}(\sin t)\ K_{n-1}^{\alpha+1}(t) - \frac{n}{\alpha+1}(\cos t - 1)[\frac{\cot(t/2)}{2} - \tilde{K}_{n-1}^{\alpha+1}(t)].$$

Proof: Multiply the recurrence in Lemma 2 by $e^{it/2}/2A_n^\alpha \sin(t/2)$ and take real and imaginary parts. Use

$$A_n^\alpha = \binom{n+\alpha}{n}; \ \frac{A_{n-1}^{\alpha+1}}{A_n^\alpha} = \frac{n}{\alpha+1}.$$

We now take up the proof of Theorem 2.

Integrate the first equation of Lemma 3 over $[-\pi, \pi]$ after multiplying by $f(x+t)/\pi$. This produces:

$$\begin{aligned} \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t) K_n^\alpha(t) dt = c_o(f) \quad &+ \quad \frac{n}{\alpha+1}\left\{\frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)(\cos t)\ K_{n-1}^{\alpha+1}(t) dt \right. \\ &- \quad \left. \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t) K_{n-1}^{\alpha+1}(t) dt \right\} \\ &+ \quad \frac{n}{\alpha+1}\left\{\frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\frac{(\sin t)\cot(t/2)}{2} dt \right. \\ &- \quad \left. \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)(\sin t)\tilde{K}_{n-1}^{\alpha+1}(t) dt \right\} \end{aligned}$$

Next, we calculate

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)(\cos t)K_{n-1}^{\alpha+1}(t)dt\} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(s)\cos(s-x)K_{n-1}^{\alpha+1}(s-x)ds$$

$$= \frac{1}{\pi}\left\{\cos x\int_{-\pi}^{\pi} f(s)(\cos s)K_{n-1}^{\alpha+1}(s-x)ds\right.$$

$$\left. + \sin x\int_{-\pi}^{\pi} f(s)(\sin s)K_{n-1}^{\alpha+1}(s-x)ds\right\}$$

$$= \cos x\ \sigma_{n-1}^{\alpha+1}(f\cos;x) + \sin x\ \sigma_{n-1}^{\alpha+1}(f\sin;x).$$

Similarly,

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\frac{(\sin t)\cot(t/2)}{2}dt = \sin x(\widetilde{f\cos})(x) - \cos x(\widetilde{f\sin})(x).$$

$$\frac{+1}{\pi}\int_{-\pi}^{\pi} f(x+t)\sin t\ \tilde{K}_{n-1}^{\alpha+1}(t)dt = \sin x\ \tilde{\sigma}_{n-1}^{\alpha+1}(f\cos;x) - \cos x\ \tilde{\sigma}_{n-1}^{\alpha+1}(f\sin;x).$$

Hence,

$$\sigma_n^{\alpha}(f;x) = c_o(f) + \frac{n}{\alpha+1}\left\{\cos x\ \sigma_{n-1}^{\alpha+1}(f\cos;x)\right.$$

$$\left. + \sin x\ \sigma_{n-1}^{\alpha+1}(f\sin;x) - \sigma_{n-1}^{\alpha+1}(f;x)\right\}$$

$$+ \frac{n}{\alpha+1}\left\{\cos x(\tilde{\sigma}_{n-1}^{\alpha+1}(f\sin;x) - (\widetilde{f\sin})(x))\right.$$

$$\left. - \sin x(\tilde{\sigma}_{n-1}^{\alpha+1}(f\cos;x) - (\widetilde{f\cos})(x))\right\}$$

The last curly bracket is $o(\frac{1}{n})$. The expression in the first curly bracket can be written

$$\cos x\left[\sigma_{n-1}^{\alpha+1}(f\cos;x) - f(x)\cos x\right]$$

$$+\sin x\left[\sigma_{n-1}^{\alpha+1}(f\sin;x) - f(x)\sin x\right]$$

$$+f(x) - \sigma_{n-1}^{\alpha+1}(f;x) = o(\tfrac{1}{n})$$

Finally, after multiplying the second equation in Lemma 3 by $f(x+t)/\pi$ and integrating over $[-\pi,\pi]$, an entirely similar argument shows that $\tilde{\sigma}_n^{\alpha}(f;x) \to 0$.

References

[1] G.H. Hardy, Divergent Series, Oxford University Press, 1949.

[2] A. Zygmund, Trigonometric Series, Cambridge University Press, 1959.

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