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CERTAIN MEASURE ZERO, FIRST CATEGORY SETS

I

P. Komjáth [4] constructed a measure zero, first category set containing a translated copy of every countable set of reals. Assuming Martin's Axiom, he generalized this by showing that a similar set exists containing a translated copy of every set of size smaller than $|R|$. In this paper we obtain some results exploring such sets. For example, we show that if a set A contains a translated copy of every countable set, then its intersection with every nonempty open interval is of size $|R|$. The converse of this result is not true.

Theorem 1. *Let A be a set of reals which contains a translated copy of every countable set of reals. If F is a measure zero or first category additive subgroup of the reals R, or if F is a set of size smaller than $|R|$, then $A \setminus F$ contains a translated copy of every countable set of reals.*

Proof. Let N be a countable set of reals and let

$$X = \{x \in R : x + N \subseteq A\}, \quad \text{where } x + N = \{x + y : y \in N\}.$$

Suppose $(x + N) \cap F \neq \emptyset$ for every $x \in X$. Then $X \subseteq F - N$ and $X - X$ is contained in a countable union of translated copies of F or $|X - X| < |R|$. Let $y \in R$ and $y \notin X - X$. The hypothesis implies that $r + ((N + y) \cup N) \subseteq A$ for some $r \in R$. Since $r + y + N \subseteq A$ and $r + N \subseteq A$, by definition of X, $r + y \in X$ and $r \in X$. Hence $y \in X - X$, which contradicts the choice of y. Thus $(x + N) \cap F = \emptyset$ for some $x \in X$ and this completes the proof.

Corollary 1. *If a set A contains a translated copy of every countable set, then its intersection with every nonempty open interval is of size $|R|$.*

Proof. If $|A \cap I| < |R|$ for some nonempty open interval I, then by Theorem 1, $A \setminus (A \cap I)$ contains a translated copy of every countable set, and consequently

$A \setminus (A \cap I)$ is everywhere dense in R. This implies that $(A \setminus (A \cap I)) \cap I \neq \emptyset$, which is impossible.

The following example shows that the converse of Corollary 1 is not true.

Example. It is well known that the group of real numbers R under addition is (group) isomorphic to the group of complex numbers C under addition. Since $C = \{(x,0) : x \in R\} + \{(0,y) : y \in R\}$ and isomorphic image of a subgroup of C is a subgroup of R, $R = A + B$, where A and B are subgroups of R, $A \cap B = \{0\}$ and $|A| = |B| = |R|$. First we note that if $a \in A$, then $\frac{m}{n}a \in A$ for every rational number $\frac{m}{n}$. [$\frac{m}{n}a = a_1 + b_1$, $a_1 \in A$ and $b_1 \in B$ imply that $ma - na_1 = nb_1 \in A \cap B = \{0\}$; hence $b_1 = 0$]. Let I be a nonempty open interval. Then for every nonzero a in A, there exists a rational number r such that $ra \in I$. Since $|A| = |R|$ and the set of all rational numbers is countable, there exists a rational number r such that $ra \in I$ for $|R|$ many a in A. Thus $|I \cap A| = |R|$. It can be easily shown that a set A contains a translated copy of every set of size 2 if and only if $A - A = R$. Since $A - A = A \neq R$, A does not contain a translated copy of every countable set.

Definition. A set $S \subseteq R$ is called a Sierpiński set if S is uncountable and $S \cap H$ is countable for every measure zero set H.

A. Miller [7] proved that if S is a Sierpiński set and $\bigcup_{n<\omega} C_n$ is a union of compact sets of measure zero, then $\{x : (x + \bigcup_{n<\omega} C_n) \cap S = \emptyset\}$ is comeager.

He asks in his paper whether the theorem is true for all measure zero sets. We prove the following theorem.

Theorem 2. *If S is any nonempty subset of R, then there exists a measure zero set M such that $\{x : (x + M) \cap S = \emptyset\}$ is meager.*

Proof. Let $y \in S$, M be any comeager set of measure zero and $X = \{x : (x + M) \cap S = \emptyset\}$. If X is not meager, then $X \cap (y - M) \neq \emptyset$ and consequently $y \in (X + M) \cap S$.

It is not known if $\{x : (x + M) \cap S = \emptyset\}$ is comeager whenever S is a Sierpinski set and M is a measure zero meager set.

Definition. A_{null} means, for every $f : R \to R_{\text{null}}$ (i.e., f assigns to each real number a Lebesgue measure zero set), there exist distinct x, y in R such that $x \notin f(y)$ and $y \notin f(x)$. $\{x, y\}$ is called a free set for f. A $_{\text{meager}}$ is defined analogously.

C. Freiling [2, p.194] asks whether the axioms A_{null} and A_{meager} together are formally inconsistent. Under $2^{\aleph_0} = \aleph_2$, we note that they are mutually inconsistent.

Theorem 3. *Assume $2^{\aleph_0} = \aleph_2$. Then A_{null} and A_{meager} are mutually inconsistent.*

Proof. A_{null} [2, p.193] implies that R is not the $\aleph_1$-union of null sets. Hence, by [6, Th. 1.1 (a)], every set of size $\aleph_1$ is meager. Let $<$ be a well ordering of R and let $f(x) = \{y : y \le x\}$. Then for any distinct x, y in R, $x \in f(y)$ or $y \in f(x)$ and consequently A_{meager} is false.

Theorem 4. *Existence of Sierpiński set of size greater than $\aleph_1$ implies that A_{null} is true and A_{meager} is false.*

Proof. Let S be a Sierpiński set of size greater than $\aleph_1$ and let G be a meager set such that $R \setminus G$ is null. Then $R = X + G$ for any subset X of S of size $\aleph_1$. In other words $R = \aleph_1$ union of meager sets and hence A_{meager} is false. However, it is easy to see from A. Hajnal's Theorem [3] "if f is a mapping from a set Y into the power set $P(Y)$ of Y such that $|f(y)| < K\ \forall\, y \in Y$, where K is a fixed cardinal smaller than $|Y|$, then there is a free set for f of size $|Y|$" that every Sierpiński set of size $\aleph_2$ contains a free set of size $\aleph_2$ for any measure zero set mapping on R, in particular A_{null} is true and the proof is complete. [Note that measure zero set mapping on R is a mapping $f : R \to P(R)$ such that $f(x)$ is of measure zero. Recall that a set $X \subseteq R$ is a free set for f if $x \notin f(y)$ and $y \notin f(x)$ for any distinct x, y in X].

P. Komjáth [4], assuming Martin's Axiom, proved that there exists a null, meager set containing a translated copy of every null, meager set of size smaller than $|R|$. Recently he [5] proved that it is consistent that $2^{\aleph_0} = \aleph_2$ and no meager set contains a translated copy of every null, meager set of size smaller than $|R|$.

Theorem 5. *$2^{\aleph_0} = \aleph_2$ and the existence of Sierpiński set of size $\aleph_2$ imply that there exists a meager set containing a translated copy of every set of size smaller than $|R|$.*

Proof. Otherwise R is an $\aleph_1$-union of null sets, but by theorem 4, the assumptions imply A_{null} and hence a contradiction.

C. Freiling [2] proved that assuming $2^{\aleph_0} = \aleph_2$, A_{null} [i.e., every measure zero mapping on R admits a free set of size 2] if and only if there exists a nonmeasurable set of size $\aleph_1$, and R is not the $\aleph_1$-union of null sets.

The following theorem is interesting in its own right.

Theorem 6. *Assume $2^{\aleph_0} = \aleph_2$. Every measure zero mapping f on R admits an infinite free set $\Leftrightarrow$ R is not the $\aleph_1$-union of sets A_i, where every subset of A_i of size $\aleph_1$ is null.*

Proof. ($\Rightarrow$) It is not hard to show that if R is the $\aleph_1$-union of sets A_i, where every subset of A_i of size $\aleph_1$ is null, then there exists a measure zero mapping on R without infinite free set. Without loss of generality we may assume that the sets A_i are disjoint and $|A_i| = |R|$. Well-order $A_i = (a_i^j)_{j<\omega_2}$ and define $f(a_i^j) = \{a_n^m : m \leq j, n \leq i\}$. Any countably infinite free set is of the form $\{a_{n_i}^{m_i}\}$, where (n_i) is an increasing sequence of ordinals. Again from the definition of f and the definition of free set, (m_i) is an infinite decreasing sequence of ordinals, which is impossible.

To prove (<=) we need the following lemma.

Lemma. *Assume* $2^{\aleph_0} = \aleph_2$. *Let* M *be a set such that* M *is not the* $\aleph_1$*-union of sets* A_i, *where every subset of* A_i *of size* $\aleph_1$ *is null* $(*)$. *Then there exists* x *in* M *such that* $M \setminus f^{-1}(x)$ *has the same property* $(*)$ *as* M, *where* $f^{-1}(x) = \{y : x \in f(y)\}$.

Proof of the Lemma. Clearly M contains a nonmeasurable set P of size $\aleph_1$. If $M \setminus f^{-1}(x)$ does not have the property $(*)$ for every x in M, then $\bigcup_{x\in P}(M \setminus f^{-1}(x)) = M \setminus \bigcap_{x\in P} f^{-1}(x) = M$ does not have the property $(*)$, which contradicts the hypothesis of the lemma. Now we use an idea of [1] to prove $(\Leftarrow)$. Let $R_1 = R \setminus (f^{-1}(x_1) \cup f(x_1) \cup \{x_1\})$. Then $\{x_1, y\}$ is a free set for any $y \in R_1$. Now the lemma ensures that the process can be repeated infinitely often. Thus the proof is complete.

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