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SOME PARAMETERS OF DISTRIBUTION OF MASS IN SELFSIMILAR FRACTALS

1.- Introduction and preliminaries

Given a fractal set one may define different parameters in order to obtain information about the spatial distribution of the mass of the fractal. Some of them are the centre of mass and the moment of inertia which we introduce in section 2.

If the fractal is selfsimilar then the selfsimilarity may be used to obtain easy formulas to compute those parameters as we show in section 3.

We shall give first some preliminaries with exact definitions of the terms used above and the results we need.

We consider a set $E \subset \mathbb{R}^n$. Given $\delta > 0$ we shall say that a family of sets $\{A_i\}$, $i = 1, 2, \ldots$ is a δ-cover of E if $E \subset \bigcup_{i=1}^{\infty} A_i$ and the diameters $d(A_i)$ of such sets satisfy $d(A_i) \leq \delta$ for every i.

Given $s \geq 0$ a positive real number we define

$$H^s_\delta(E) = \inf \left\{ \sum_{i=1}^{\infty} d(A_i)^s \ : \ \{A_i\} \text{ is a } \delta-\text{cover of } E \right\}$$

and

$$H^s(E) = \lim_{\delta \to 0} H^s_\delta(E)$$

H^s is an outer measure which determines a Borel measure which is called the *s-dimensional Hausdorff measure.*

Given $E \subset \mathbb{R}^n$ there exists a unique number s such that

$$\begin{array}{lll} H^t(E) = 0 & \text{for every} & t > s \\ H^t(E) = \infty & \text{for every} & t < s \end{array}$$

Such number s is called the *Hausdorff dimension* of E.

If $0 < H^s(E) < \infty$ we shall say that E is an *s-fractal.*

A mapping $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is said to be a similarity if there exists r, $0 < r < 1$ such that

$$|\varphi(x) - \varphi(y)| = r|x - y|$$

The number r is called the similarity ratio of φ.

We shall say that $E \subset \mathbb{R}^n$ is a *self-similar fractal* if there is a set $\mathcal{S} = \{\varphi_1, ..., \varphi_m\}$ of similarities such that:

(a) $E = \bigcup_{i=1}^m \varphi_i(E)$

(b) E is H^s-measurable for some s, $0 < s \leq n$, and

(b1) $0 < H^s(E) < \infty$

(b2) $H^s(\varphi_i(E) \cap \varphi_j(E)) = 0$, for $i \neq j$, $i, j = 1, ..., m$.

Given a set $\mathcal{S} = \{\varphi_1, ..., \varphi_m\}$ of similarities we say that it satisfies the *open set condition* if there is a bounded open set $V \subset \mathbb{R}^n$, $V \neq \emptyset$, such that

$$S(V) = \bigcup_{i=1}^{m} \varphi_i(V) \subset V$$

and for $i \neq j$, $i, j = 1, ..., m$,

$$\varphi_i(V) \cap \varphi_j(V) = \emptyset$$

If the open set condition holds, then $0 < H^s(E) < \infty$ where s is the real number which satisfies $\sum_{i=1}^m r_i^s = 1$, where r_i is the similarity ratio of φ_i. Then, the Hausdorff dimension of E is s (see [H]). For more details and proofs see [FK].

2.- Centre of mass and moment of inertia of a fractal set

The following concepts inform us about the distribution of mass of the fractal set.

Definition 2.1 Let $E \subset \mathbb{R}^n$ be a fractal set such that $0 < H^s(E) < \infty$. The *centre of mass* of E is the point $c = (c_k) \in \mathbb{R}^n$, $k = 1, ..., n$, whose coordinates c_k are

$$c_k = \frac{\int_E x_k \, dH^s(x)}{H^s(E)} \quad , \quad x = (x_1, ..., x_n) \quad , \quad k = 1, ..., n$$

Definition 2.2 Let $E \subset \mathbb{R}^n$ be such that $0 < H^s(E) < \infty$. The *moment of inertia* of E respect to $a \in \mathbb{R}^n$ is the positive real number given by

$$I_a(E) = \int_E |a - x|^2 \, dH^s(x)$$

In a similar way as in the classic theorem of Steiner one can obtain the following result.

Theorem 2.3

Let $E \subset \mathbb{R}^n$ such that $0 < H^s(E) < \infty$ and $a, c \in \mathbb{R}^n$, then

$$I_a(E) = I_c(E) + H^s(E) \cdot |a - c|^2$$

3.- Formulas for centre of mass and moment of inertia of selfsimilar fractals

For selfsimilar fractals we can obtain formulas to find the elements defined above.

Theorem 3.1

Let $E \subset \mathbb{R}^n$ be a selfsimilar fractal associated to the family of similarities $\mathcal{S} = \{\varphi_1, ..., \varphi_m\}$ with similarity ratios $\{r_1, ..., r_m\}$ respectively. Suppose the set $\mathcal{S}$ verifies the open set condition, and $c \in \mathbb{R}^n$ is the centre of mass of E. Then

(a) $c_k = \sum_{i=1}^m r_i^s (\varphi_i(c))_k$, for $k = 1, ..., m$.

(b) $I_c(E) = \sum_{i=1}^m [r_i^{s+2} I_c(E) + r_i^s H^s(E) |c - \varphi_i(c)|^2]$

Proof

(a) Under our hypothesis we know that $0 < H^s(E) < \infty$ where s is the number such that $\sum_{i=1}^m r_i^s = 1$.

Let $d_i = (d_i)_k$ be the centre of mass of $\varphi_i(E)$ and $c = (c_k)$ the centre of mass of E, $(k = 1, ..., n)$.

Since $\varphi_i : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a similarity then

$$\varphi_i(t) = A_i t + B_i$$

where A_i and B_i are constant matrixes of order $n \times n$ and $n \times 1$ respectively.

From $|\varphi_i(x) - \varphi_i(y)| = r_i|x - y|$, $0 < r_i < 1$, for $i = 1, ..., m$, we can easily show, as a consequence of the definition of the measure H^s, that

$$H^s[\varphi_i(E \cap B)] = r_i^s \cdot H^s(E \cap B) \tag{1}$$

for $i = 1, ..., m$ and every Borel set B. Then by definition 2.1 and the change of variable formula

$$\begin{aligned} d_i &= \frac{1}{H^s(\varphi_i(E))}\left(\int_{\varphi_i(E)} x_1\, dH^s(x), ..., \int_{\varphi_i(E)} x_n\, dH^s(x)\right) \\ &= \frac{1}{r_i^s H^s(E)}\left(\int_E r_i^s(\varphi_i(t))_1 dH^s(t), ..., \int_E r_i^s(\varphi_i(t))_n dH^s(t)\right) \\ &= \left(\int_E (A_i t + B_i)_1 dH^s(t), ..., \int_E (A_i t + B_i)_n dH^s(t)\right) \\ &= A_i\left(\int_E x_1\, dH^s(x), ..., \int_E x_n\, dH^s(x)\right) + B_i \end{aligned}$$

where the last equality may be obtained by routinary computations and is a consequence of the linearity of the integral.

So we have

$$d_i = \varphi_i(c) \qquad \text{and} \qquad (d_i)_k = (\varphi_i(c))_k \tag{2}$$

for $k = 1, ..., n$ and $i = 1, ..., m$, result which will be applied later.

Taking into account the properties

$$E = \bigcup_{i=1}^{m} \varphi_i(E) \quad , \quad \text{and} \quad H^s(\varphi_i(E) \cap \varphi_j(E)) = 0 \quad \text{for } i \neq j \tag{3}$$

contained in the definition of selfsimilar fractal, then

$$\begin{aligned} c_k &= \frac{1}{H^s(E)}\int_E x_k\, dH^s(x) = \frac{1}{H^s(E)}\int_{\bigcup_{i=1}^m \varphi_i(E)} x_k\, dH^s(x) \\ &= \frac{1}{H^s(E)}\sum_{i=1}^{m}\int_{\varphi_i(E)} x_k\, dH^s(x) = \frac{1}{H^s(E)}\sum_{i=1}^{m} H^s(\varphi_i(E))(d_i)_k \end{aligned}$$

Finally, by replacing the value of $(d_i)_k$ obtained in (2) and the fact

$$H^s(\varphi_i(E)) = r_i^s H^s(E) \tag{4}$$

which is a consequence of (1), we obtain

$$c_k = \sum_{i=1}^{m} r_i^s(\varphi_i(c))_k \tag{5}$$

formula which may be used to obtain the coordinates of the centre of mass of E when we know the similarities φ_i.

(b) From definition 2.2 and (3) we may write

$$\begin{aligned} I_c(E) &= \int_E |x-c|^2 dH^s(x) = \int_{\bigcup_{i=1}^m \varphi_i(E)} |x-c|^2 dH^s(x) \\ &= \sum_{i=1}^{m} I_c(\varphi_i(E)) \end{aligned}$$

On the other hand, from theorem 2.3 we have

$$I_c(\varphi_i(E)) = I_{\varphi_i(c)}(\varphi_i(E)) + H^s(\varphi_i(E))|c-\varphi_i(c)|^2 \quad \text{for} \quad i = 1, ..., m$$

and then

$$I_c(E) = \sum_{i=1}^{m} I_c(\varphi_i(E)) = \sum_{i=1}^{m} \left(I_{\varphi_i(c)}(\varphi_i(E)) + H^s(\varphi_i(E))|c-\varphi_i(c)|^2\right) \tag{6}$$

Let us now find $I_{\varphi_i(c)}(\varphi_i(E))$.

By definition 2.2 and making a change of variable

$$I_{\varphi_i(c)}(\varphi_i(E)) = \int_{\varphi_i(E)} |\varphi_i(c) - x|^2 dH^s(x) = r_i^s \int_E |\varphi_i(c) - \varphi_i(x)|^2 dH^s(x)$$

But since $|\varphi_i(c) - \varphi_i(x)| = r_i|c-x|$, then

$$I_{\varphi_i(c)}(\varphi_i(E)) = r_i^{s+2} \int_E |c-x|^2 dH^s(x) = r_i^{s+2} I_c(E) \tag{7}$$

Combining (6), (7) and (4), we get

$$I_c(E) = \sum_{i=1}^{m} \left(r_i^{s+2} I_c(E) + r_i^s H^s(E)|c-\varphi_i(c)|^2\right)$$

formula which permit us to know the moments of inertia of the selfsimilar fractal E by mean of the similarities $\varphi_1, ..., \varphi_m$ which determine it.

References

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