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LIMITS WITHOUT EPSILONS

The concept of convergence of real sequences is completely characterized by six properties, each essentially a theorem from the theory of limits of sequences of real numbers. Consequently, the foundation of the calculus can be constructed without the use of the "ϵ, δ" definitions of Cauchy. The six properties are versions of: (1) the scalar multiple of a convergent sequence is convergent; (2) the corresponding sums of equivalent convergent sequences are equivalent; (3) the Squeezing Theorem; (4) all subsequences of a convergent sequence converge and are equivalent; (5) the sequence $1, -1, 1, -1, \cdots$ does not converge; and (6) every divergent and bounded sequence has two non-equivalent, convergent subsequences. Five of the six properties are shown to be necessary for the characterization of convergence. The question of the independence of the Squeezing Theorem from the other five properties remains unresolved.

INTRODUCTION: The following characterization of the convergence of real sequences does not use the classical definitions of limits, first used by Cauchy [3] in the nineteenth century. Furthermore, the setting of the real number system remains the same. There are no unusual or unorthodox changes to the structure of the calculus or real analysis. Each of the conditions contained in this characterization of limits of sequences has an analog in the classical theory of limits of sequences.

The characterization is shown to be precisely equivalent to the concept, "limit of a real sequence," as typically defined and universally accepted. In other words, any theorem about the convergence of sequences of real numbers can be proven in this setting. With the exception of one condition, the characterization is shown to be minimal in the sense that each condition is necessary to preserve the uniqueness of the characterization.

The author purposely developed these concepts in the context of the real number system for two reasons. The first reason was to show that this approach is a practical alternative to the standard method using the Cauchy type(ϵ, δ) definitions of convergence, traditionally used in elementary calculus and real analysis. The second reason is to present these six properties as an axiom system in the setting used in the nineteenth century for the Cauchy convergence definitions. This system can and should be generalized in several directions. The comparable concepts of limits of functions and continuity can be examined in a manner analogous to the approach given here. The investigation of generalizations and similar developments should definitely be explored at a later time.

It is surprising that this system was not developed fifty to one hundred years ago. Nevertheless, the author has not found any reference to a system that considers the very common six properties collectively as a basis for convergence, despite considerable searching in the literature and consulting with many colleagues.

DEFINITIONS: It is assumed that the real number system considered here does not use the "epsilon" type conditions of Cauchy in its definition. The collection of real sequences will be denoted by S. The set of convergent sequences of S will be denoted by C. $(A, S, \equiv)$ is a *convergence system on* S will mean that $A \subseteq S$, " $\equiv$ " is a relation on S and:

A1. If $X = \{x_n\}_{n=1}^{\infty} \in A$ and $k \in R$(reals), then $kX = \{kx_n\}_{n=1}^{\infty} \in A$;

A2. If X, Y, Z, W, $X + Z$, $Y + W \in A$, $X \equiv Y$ and $Z \equiv W$, then $Y + W \equiv X + Z$;

A3. If $X, Y \in A$, $Z \in S$, $\forall n$, $x_n \leq z_n \leq y_n$ and $X \equiv Y$, then $Z \in A$ and $Z \equiv X$;

A4. If $X \in A$ and Y, Z are subsequences of X, then $Y, Z \in A$ and $Y \equiv Z$;

A5. $\{(-1)^n\}_{n=1}^{\infty} \notin A$; and

A6. If $X \notin A$ and X is bounded, then X has two subsequences $Y, Z \in A$ such that $Y \not\equiv Z$.

CONVERGENCE SYSTEMS: This section establishes some basic properties of convergence systems that are used later to prove that the concept of convergence systems completely characterizes the notion of "limit of real sequences."

Assume that $(A, S, \equiv)$ is an arbitrary convergence system in Theorems 1-7.

Theorem 1: If $a \in R$, then $\{a\}_{n=1}^{\infty} \in A$.

Proof: Suppose $X = \{a\}_{n=1}^{\infty} \notin A$. By A6, there exists two subsequences $Y, Z \in A$ of X such that $Y \not\equiv Z$. Since X is constant, $Y = Z = X$ and $X \in A$. Contradiction.

Theorem 2: $\{1\}_{n=1}^{\infty} \not\equiv \{0\}_{n=1}^{\infty}$.

Proof: Suppose $\{1\}_{n=1}^{\infty} \equiv \{0\}_{n=1}^{\infty}$. $\{-1/2\}_{n=1}^{\infty}$ is a subsequence of itself.
As a result, $\{-1/2\}_{n=1}^{\infty} \equiv \{-1/2\}_{n=1}^{\infty}$ and
$\{-1/2\}_{n=1}^{\infty} = \{0\}_{n=1}^{\infty} + \{-1/2\}_{n=1}^{\infty} \equiv \{1\}_{n=1}^{\infty} + \{-1/2\}_{n=1}^{\infty} = \{1/2\}_{n=1}^{\infty}$. For every positive integer n, $-1/2 \leq (-1)^n/2 \leq 1/2$. Consequently, $\{(-1)^n/2\}_{n=1}^{\infty} \in A$ and $\{(-1)^n\}_{n=1}^{\infty} \in A$. Contradiction.

Theorem 3: If $X \in A$, then X is bounded.

Proof: Suppose $X \in A$ and X is unbounded above. Let Y denote a subsequence of X such that for every n, $y_n \leq y_n + 1/2 \leq y_{n+1}$.
The sequences $\{y_n\}_{n=1}^{\infty}, \{y_{n+1}\}_{n=1}^{\infty}, \{-y_n\}_{n=1}^{\infty} \in A$ and $\{y_n\}_{n=1}^{\infty} \equiv \{y_{n+1}\}_{n=1}^{\infty}$.
As a result, $\{y_n + 1/2\}_{n=1}^{\infty} \in A$ and $\{y_n + 1/2\}_{n=1}^{\infty} \equiv \{y_n\}_{n=1}^{\infty}$. $\{-y_n\}_{n=1}^{\infty}$ is a subsequence of itself. Consequently, $\{-y_n\}_{n=1}^{\infty} \equiv \{-y_n\}_{n=1}^{\infty}$. By A2, it

follows that, $\{0\}_{n=1}^{\infty} = \{y_n\}_{n=1}^{\infty} + \{-y_n\}_{n=1}^{\infty} \equiv \{y_n + 1/2\}_{n=1}^{\infty} + \{-y_n\}_{n=1}^{\infty} = \{1/2\}_{n=1}^{\infty}$. As a result, $\{1\}_{n=1}^{\infty} = \{1/2\}_{n=1}^{\infty} + \{1/2\}_{n=1}^{\infty} \equiv \{0\}_{n=1}^{\infty} + \{0\}_{n=1}^{\infty} = \{0\}_{n=1}^{\infty}$. Contradiction. A similar argument holds when X is unbounded below. Consequently, X is bounded.

Theorem 4: If $X, Y \in A$, then $X + Y \in A$.

Proof: Suppose $X, Y \in A$ and $X + Y \notin A$. $X + Y$ is bounded. By A6, there exist two subsequences $Z, W \in A$ of $X + Y$ such that $Z \not\equiv W$. There exist two subsequences $U, V \in A$ of X and two subsequences $S, T \in A$ of Y such that $U + S = Z$ and $V + T = W$. Since $V \equiv U$ and $T \equiv S$, $Z = U + S \equiv V + T = W$. Contradiction.

Theorem 5: The relation " $\equiv$ " is an equivalence relation on A.

Proof: (Reflexive) Suppose $X \in A$. X is a subsequence of X. By A4, $X \equiv X$.
(Symmetric) Suppose $X, Y \in A$ and $X \equiv Y$. $\{0\}_{n=1}^{\infty} \equiv \{0\}_{n=1}^{\infty}$.
It follows from A2 that $Y = Y + \{0\}_{n=1}^{\infty} \equiv X + \{0\}_{n=1}^{\infty} = X$.
(Transitive) Suppose $X, Y, Z \in A$, $X \equiv Y$ and $Y \equiv Z$.
Again using A2, $Z + Y,\ Y + X \in A$ and $Z + Y \equiv Y + X$.
Since $-Y \in A$ and $-Y \equiv -Y$, $X = (Y + X) + (-Y) \equiv (Z + Y) + (-Y) = Z$.

Theorem 6: $\{1/n\}_{n=1}^{\infty}, \{-1/n\}_{n=1}^{\infty}, \{2/n\}_{n=1}^{\infty} \in A$ and
$\{1/n\}_{n=1}^{\infty} \equiv \{0\}_{n=1}^{\infty} \equiv \{-1/n\}_{n=1}^{\infty} \equiv \{2/n\}_{n=1}^{\infty}$.

Proof: Suppose $\{1/n\}_{n=1}^{\infty} = X \notin A$. There exist two subsequences $Y, Z \in A$ of X such that $Y \not\equiv Z$. There exists a subsequence W of Y and a subsequence V of Z such that for every positive integer n, $w_{n+1} \le v_n \le w_n$. Since $\{w_{n+1}\}_{n=1}^{\infty}$ and $\{w_n\}_{n=1}^{\infty}$ are subsequences of Y, $\{w_{n+1}\}_{n=1}^{\infty} \equiv \{w_n\}_{n=1}^{\infty}$. By A3, $\{v_n\}_{n=1}^{\infty} \in A$ and $V \equiv W$. Consequently, $Y \equiv W \equiv V \equiv Z$. Contradiction.
Therefore, $\{1/n\}_{n=1}^{\infty}, \{-1/n\}_{n=1}^{\infty}, \{2/n\}_{n=1}^{\infty} \in A$. $\{2/n\}_{n=1}^{\infty}$ is a subsequence of $\{1/n\}_{n=1}^{\infty}$ and $\{1/n\}_{n=1}^{\infty} \equiv \{2/n\}_{n=1}^{\infty}$. Since $\{-1/n\}_{n=1}^{\infty} \equiv \{-1/n\}_{n=1}^{\infty}$, $\{1/n\}_{n=1}^{\infty} = \{2/n\}_{n=1}^{\infty} + \{-1/n\}_{n=1}^{\infty} \equiv \{1/n\}_{n=1}^{\infty} + \{-1/n\}_{n=1}^{\infty} = \{0\}_{n=1}^{\infty}$, and $\{-1/n\}_{n=1}^{\infty} = \{0\}_{n=1}^{\infty} + \{-1/n\}_{n=1}^{\infty} \equiv \{1/n\}_{n=1}^{\infty} + \{-1/n\}_{n=1}^{\infty} = \{0\}_{n=1}^{\infty}$.

Theorem 7: If $X, Y \in A$ and $X \equiv Y$, then for every positive integer p, $pX = \{px_n\}_{n=1}^{\infty} \equiv \{py_n\}_{n=1}^{\infty} = pY$.

Proof: The result follows from A2 and induction.

CHARACTERIZATION OF LIMITS: The following theorems show that convergence systems exist, the set of "convergent sequences" is unique and the relation described in the definition of convergence systems is unique.

Theorem 8: $(C, S, \equiv)$ is a convergence system on S where $X \equiv Y$ means that X and Y are equivalent Cauchy sequences.

Proof: Each of the six conditions for a convergence system, with C replacing A and $X \equiv Y$ meaning X and Y are equivalent Cauchy sequences, is an elementary result for limits of sequences.

Theorem 9: If $(A, S, \equiv)$ is a convergence system on S, then $A \subseteq C$.

Proof: Suppose $X \in A$ and $X \notin C$. By the standard definition of convergence, there is a positive integer p such that for every positive number K there are positive integers $m, n \geq K$ such that $|x_n - x_m| \geq 1/p$. Let Y, Z denote two subsequences of X such that for every positive integer k, $y_k - z_k \geq 1/p$. Therefore, $z_k \leq z_k + 1/p \leq y_k$. Since $Y, Z \in A$ and $Y \equiv Z$, $\{z_n + 1/p\}_{n=1}^{\infty} \in A$ and $Z = \{z_n\}_{n=1}^{\infty} \equiv \{z_n+1/p\}_{n=1}^{\infty}$. It follows that $\{1/p\}_{n=1}^{\infty} = \{(z_n+1/p)-z_n\}_{n=1}^{\infty} = \{z_n+1/p\}_{n=1}^{\infty}-\{z_n\}_{n=1}^{\infty}$, $\{z_n+1/p\}_{n=1}^{\infty}-\{z_n\}_{n=1}^{\infty} \equiv \{z_n\}_{n=1}^{\infty}-\{z_n\}_{n=1}^{\infty} = \{0\}_{n=1}^{\infty}$, and $\{1/p\}_{n=1}^{\infty} \equiv \{0\}_{n=1}^{\infty}$. Consequently, $\{1\}_{n=1}^{\infty} = \{p/p\}_{n=1}^{\infty} = p\{1/p\}_{n=1}^{\infty} \equiv p\{0\}_{n=1}^{\infty} = \{p0\}_{n=1}^{\infty} = \{0\}_{n=1}^{\infty}$. Contradiction.

Theorem 10: If $(A, S, \equiv)$ is a convergence system on S, then $C \subseteq A$.

Proof: Suppose $X \in C$ and $X \notin A$. Therefore, X is bounded. Let $Y, Z \in A$ denote two subsequences of X such that $Y \not\equiv Z$. Let W denote a subsequence of X such that $\{w_{2n}\}_{n=1}^{\infty}$ is a subsequence of Y, $\{w_{2n+1}\}_{n=1}^{\infty}$ is a subsequence of Z and for every $n \in N$, $-1/n \leq w_{2n} - w_{2n+1} \leq 1/n$. Since $\{1/n\}_{n=1}^{\infty} \equiv \{-1/n\}_{n=1}^{\infty}$, $\{w_{2n} - w_{2n+1}\}_{n=1}^{\infty} \in A$ and $\{w_{2n} - w_{2n+1}\}_{n=1}^{\infty} \equiv \{1/n\}_{n=1}^{\infty} \equiv \{0\}_{n=1}^{\infty}$.

Y$\equiv$ $\{w_{2n}\}_{n=1}^{\infty}$ and $-Z \equiv \{-w_{2n+1}\}_{n=1}^{\infty}$, since $\{w_{2n}\}_{n=1}^{\infty}$ and $\{w_{2n+1}\}_{n=1}^{\infty}$ are subsequences of Y and Z, respectively. As a result, $Y - Z \equiv \{w_{2n}\}_{n=1}^{\infty} - \{w_{2n+1}\}_{n=1}^{\infty} = \{w_{2n}-w_{2n+1}\}_{n=1}^{\infty} \equiv \{0\}_{n=1}^{\infty}$, and $Y = Z+(Y-Z) \equiv Z+\{0\}_{n=1}^{\infty} = Z$. Consequently, $Y \equiv Z$. Contradiction.

Theorem 11: If $(A, S, \equiv)$ is a convergence system on S and $(A, S, \cong)$ is a convergence system on S, then " $\equiv$ " and " $\cong$ " are the same relation on A.

Proof: Suppose $X, Y \in A$ and $X \equiv Y$. By A2, $X - Y \equiv Y - Y = \{0\}_{n=1}^{\infty}$, $\{0\}_{n=1}^{\infty} \equiv \{1/n\}_{n=1}^{\infty}$, $X - Y \equiv \{1/n\}_{n=1}^{\infty}$. Suppose there is a positive integer p and a subsequence W of $X - Y$ such that for every $n \in N$, $0 \leq 1/p \leq w_n$. As a result, $W \equiv \{0\}_{n=1}^{\infty}$, $\{0\}_{n=1}^{\infty} \equiv \{1/p\}_{n=1}^{\infty}$ and $\{0\}_{n=1}^{\infty} = \{p0\}_{n=1}^{\infty} = p\{0\}_{n=1}^{\infty} \equiv p\{1/p\}_{n=1}^{\infty} = \{1\}_{n=1}^{\infty}$. Contradiction.

A similar argument holds for the case when there is a positive integer p and a subsequence W of $X - Y$ such that for every $n \in N$, $w_n \leq -1/p \leq 0$. Therefore, there is a subsequence W of $X-Y$ such that for every $n \in N$, $-1/n \leq w_n \leq 1/n$. Since $\{-1/n\}_{n=1}^{\infty} \cong \{1/n\}_{n=1}^{\infty}$, $X - Y \cong W \cong \{1/n\}_{n=1}^{\infty} \cong \{0\}_{n=1}^{\infty}$. Consequently, $X \cong Y$. Likewise when $X, Y \in A$ and $X \cong Y$.

Since Theorem 8 shows that ordinary convergence is an example of a convergence system and Theorems 9-11 show uniqueness of convergence systems, the classical concept of convergence of real sequences is completely characterized by convergence systems.

MUTUAL INDEPENDENCE OF CONVERGENCE CONDITIONS:

The following examples show, with one exception, that each of the conditions in the definition of a convergence system is necessary in order to preserve uniqueness. In other words, any five of the conditions taken as a system has multiple examples. The one exception is the "Squeezing Theorem" or condition A3. The question of the dependence or independence of this condition with respect to the other five conditions is unresolved.

Example 1: The following is an example of a system $(E1, S, \equiv)$ that satisfies conditions A2 through A6 of the definition of a convergence system, but does not satisfy condition A1.

Let $E1 = C \cup Dn$ where Dn denotes the set of real sequences that diverge downward (all subsequences diverge downward). Let $X \equiv Y$ mean that X and Y are equivalent Cauchy sequences when $X, Y \in C$, let $X \equiv Y$ when $X, Y \in Dn$ and let $X \not\equiv Y$ otherwise.

Proof that A1 does not hold: $\{-n\}_{n=1}^{\infty} \in E1$ and $(-1)\{-n\}_{n=1}^{\infty} = \{n\}_{n=1}^{\infty} \notin E1$.

Proof that A2 holds: Suppose $X, Y, Z, W,\ X+Z,\ Y+W \in E1$, $X \equiv Y$ and $Z \equiv W$. If $X, Y, Z, W \in C$, then $Y+W \equiv X+Z$. If $X, Y \in Dn$ and $Z, W \in Dn$, then $X+Z,\ Y+W \in Dn$ and $Y+W \equiv X+Z$. If $X, Y \in C$ and $Z, W \in Dn$, then $X+Z,\ Y+W \in Dn$ and $Y+W \equiv X+Z$.

Proof that A3 holds: Suppose $X, Y \in E1$, $Z \in S$, $\forall n,\ x_n \leq z_n \leq y_n$ and $X \equiv Y$. If $X \in C$, then $Y \in C$, $Z \in C$ and $Z \equiv X$. If $X \in Dn$, then $Y \in Dn$, $Z \in Dn$ and $Z \equiv X$.

Proof that A4 holds: Suppose $X \in E1$ and Y, Z are subsequences of X. If $X \in C$, then $Y, Z \in C$ and $Y \equiv Z$. If $X \in Dn$, then $Y, Z \in Dn$. Therefore, $Y, Z \in E1$ and $Y \equiv Z$.

Proof that A5 holds: $\{(-1)^n\}_{n=1}^{\infty} \notin C$ and $\{(-1)^n\}_{n=1}^{\infty} \notin Dn$. Consequently, $\{(-1)^n\}_{n=1}^{\infty} \notin E1$.

Proof that A6 holds: Suppose $X \notin E1$ and X is bounded. Therefore, $X \notin C$ and X has two subsequences $Y, Z \in C \subseteq E1$ such that $Y \not\equiv Z$.

Example 2: The following is an example of a system $(E2, S, \equiv)$ that satisfies conditions A1 and A3 through A6 of the definition of a convergence system, but does not satisfy condition A2.

Let $E2$ denote the set of convergent real sequences, each sequence of which does not contain two subsequences, one of which is strictly increasing and the other is strictly decreasing. Let $X \equiv Y$ mean that $X, Y \in E2$, X and Y are equivalent Cauchy sequences and if one of the sequences contains a strictly increasing subsequence, then the other sequence does not contain a strictly decreasing subsequence.

Proof that A1 holds: Suppose $X = \{x_n\}_{n=1}^{\infty} \in E2$ and $k \in R$(reals). If $k = 0$ and $X \in E2$, then $kX \in E2$. Suppose $k \neq 0$, $X \in E2$, Y is a strictly increasing subsequence of kX and Z is strictly decreasing subsequence of kX. Therefore, $(1/k)Y$ and $(1/k)Z$ are two subsequences of X such that one is strictly increasing and the other is strictly decreasing. Contradiction. Consequently, $kX \in E2$.

Proof that A2 does not hold: Let $X = \{1/n\}_{n=1}^{\infty}$, $Y = \{(1/n)^2\}_{n=1}^{\infty}$, $Z = \{-(1/n)^2\}_{n=1}^{\infty}$, and $W = \{-1/n\}_{n=1}^{\infty}$. Therefore, $X \equiv Y$, $Z \equiv W$ and $X + Z, Y + W \in E2$. The sequence $X + Z = \{1/n - (1/n)^2\}_{n=1}^{\infty}$ contains a strictly decreasing subsequence and the sequence $Y + W = \{(1/n)^2 - 1/n\}_{n=1}^{\infty}$ contains a strictly increasing subsequence. Consequently, $Y + W \not\equiv X + Z$.

Proof that A3 holds: Suppose $X, Y \in E2$, $Z \in S$, $\forall n$, $x_n \leq z_n \leq y_n$ and $X \equiv Y$. Therefore, $Z \in E2$. If one of the sequences X and Y contains a strictly increasing subsequence, then the other sequence does not contain a strictly decreasing subsequence. Consequently, if one of the sequences X and Z contains a strictly increasing subsequence, then the other sequence does not contain a strictly decreasing subsequence. In either case, $Z \equiv X$.

Proof that A4 holds: Suppose $X \in E2$ and Y, Z are subsequences of X. If one of the subsequences contains a strictly increasing subsequence, then the other subsequence does not contain a strictly decreasing subsequence. Therefore, $Y, Z \in E2$ and $Y \equiv Z$.

Proof that A5 holds: $\{(-1)^n\}_{n=1}^{\infty} \notin C$ and $E2 \subseteq C$.

Proof that A6 holds: If $X \notin E2$ and X is bounded, then $X \in C$ or $X \notin C$. If $X \in C$, then X has two subsequences $Y, Z \in E2$ such that one is strictly increasing and the other is strictly decreasing. Consequently, $Y \not\equiv Z$. If $X \notin C$, then X has two monotone subsequences $Y, Z \in E2 \subseteq C$ such that $Y \not\equiv Z$.

Example 3: The following is an example of a system $(E3, S, \equiv)$ that satisfies conditions A1, A2 and A4 through A6 of the definition of a convergence system, but does not satisfy condition A3.
The dependence or independence of condition A3 with respect to the other five conditions is unresolved.

Example 4: The following is an example of a system $(E4, S, \equiv)$ that satisfies conditions A1 through A3, A5 and A6 of the definition of a convergence system, but does not satisfy condition A4.
Let *E4* denote the set of convergent real sequences.
Let $X \equiv Y$ mean that $X = Y$.

Proof that A1 holds: Suppose $X = \{x_n\}_{n=1}^{\infty} \in E4$ and $k \in R$ (reals). Therefore, $kX \in E4$.

Proof that A2 holds: Suppose $X, Y, Z, W,\ X+Z,\ Y+W \in E4$, $X \equiv Y$ and $Z \equiv W$. Therefore, $X = Y$ and $Z = W$. Consequently, $Y + W = X + Z$ and $Y + W \equiv X + Z$.

Proof that A3 holds: Suppose $X, Y \in E4$, $Z \in S$, $\forall n,\ x_n \leq z_n \leq y_n$ and $X \equiv Y$. Therefore, $\forall n,\ x_n = z_n = y_n$ and $Z = X$. Consequently, $Z \in E4$ and $Z \equiv X$.

Proof that A4 does not hold: Suppose $X \in E4$, X is non-constant and Y is a subsequence of X such that $X \neq Y$, then $Y \in E4$, but $X \not\equiv Y$.

Proof that A5 holds: $\{(-1)^n\}_{n=1}^{\infty} \notin E4$.

Proof that A6 holds: Suppose $X \notin E4$ and X is bounded. Therefore, $X \notin C$ and X has two subsequences $Y, Z \in C \subseteq E4$ such that $Y \neq Z$. Consequently, $Y \not\equiv Z$.

Example 5: The following is an example of a system $(E5, S, \equiv)$ that satisfies conditions A1 through A4 and A6 of the definition of a convergence system, but does not satisfy condition A5.
Let $E5 = S$.
Let $X \equiv Y$ mean that $X, Y \in S$.

Conditions A1 through A4 hold since each of their conclusions is always true. Condition A5 is false since $\{(-1)^n\}_{n=1}^{\infty} \in S = E5$. Condition A6 is true since $E5 = S$, which makes the antecedent of A6 always false.

Example 6: The following is an example of a system $(E6, S, \equiv)$ that satisfies conditions A1 through A5 of the definition of a convergence system, but does not satisfy condition A6.
Let $E6 = \Phi$.
Let $X \equiv Y$ mean that $X, Y \in S$.

Conditions A1 through A4 hold since each of their antecedents is always false. Condition A5 holds since $\{(-1)^n\}_{n=1}^{\infty} \notin \Phi$. Condition A6 is false since the consequent is always false.

Conclusion: The foundation of the calculus has evolved slowly during the past three centuries. In the seventeenth century, Newton and Leibniz [3] each unified the fundamental concepts of the calculus, including the notion of the limit of a sequence of real numbers. In the nineteenth century, Cauchy [2] and others refined the concepts and gave the first modern, precise and rigorous definitions of the limit concepts. In 1960, Abraham Robinson [6] redesigned the real number system with his hyperreal numbers in order to simplify the definitions of limits. In this century, the teaching of the foundation of the calculus has used the "ϵ, δ" definitions of Cauchy [1] that had their basis in Newton's ideas. Robinson's methods have not been widely used.

The "convergence systems" characterization of limits of sequences of real numbers given here is precise and rigorous and, furthermore, it uses the universally accepted real number system. The conditions of a convergence system can be examined individually and seen to be common properties used regularly by practitioners of the calculus and real analysis. The utility of this approach to the foundation of convergence remains to be determined. Nevertheless, the convergence system defined herein is a legitimate alternative to the established definition of Cauchy.

The identification of a value for the "limit" of a sequence in a convergence system can be done in the following manner. Each equivalence class of sequences will contain exactly one constant sequence. This constant value can be defined as the limit.

Since the concepts of continuity and limits of real functions with real domains can be defined in terms of limits of sequences of real numbers [7], the foundation of the calculus can be completely based upon the concept of "convergence systems" without using the "ϵ, δ" definitions of Cauchy.

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