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ON A PROBLEM OF SKVORTSOV INVOLVING THE PERRON INTEGRAL

Skvortsov [1; p.202] posed the following problem: Prove (or disprove) that the Perron integral, defined for an abstract derivative base by means of its major and minor functions continuous with respect to this base, is equivalent to the integral for the same base but defined without the precondition of continuity of major and minor functions. In this note, we shall give a partial answer for the case of ordinary full covers [4].

First, we define the Perron integral. Let E be a compact interval in the n-dimensional Euclidean space R^n. For any $x \in R^n$ and any positive number η let $S(x, \eta)$ be the open ball with centre x and radius η. The symbol δ will always denote a positive function on E. A division $D = \{(I, x)\}$ of E is said to be δ-fine if $x \in I \subset S(x, \delta(x))$ for any interval-point pair (I, x) in D. (See [2] or [3] for further details.) An interval function A is said to be δ-superadditive if $A(I_0) \geqq (D) \sum A(I)$ for any interval $I_0 \subset E$ and any δ-fine division $D = \{(I, x)\}$ of I_0. Similarly, A is said to be δ-subadditive if $-A$ is δ-superadditive. The lower derivative of A at a point $x \in E$, denoted by $\underline{D}A(x)$, is defined as

$$\sup_{\eta>0} \inf\{A(I)/|I|;\ \ x \in I \subset S(x, \eta) \cap E)\}.$$

Let f be a function on E. Then A is called a major function of f on E if

1. A is δ-superadditive for some δ,
2. $\underline{D}A(x) \geqq f(x)$ for every $x \in E$.

Similarly, we call B a minor function of f on E if $-B$ is a major function of $-f$ on E. Finally, f is said to be (P) integrable to c on E if

$$(*) \qquad \inf A(E) = \sup B(E) = c$$

is finite, where the infimum is taken over all major functions A and the supremum over all minor functions B.

A function f defined on E is said to be (P_0) integrable if for every $\varepsilon > 0$ there exists δ, a major function A and a minor function B of f on E such that $0 \leqq A(E) - B(E) < \varepsilon$ and that for any $x \in E$ and any interval $I \subset S(x, \delta(x)) \cap E$ we have $|A(I)| < \varepsilon$ and $|B(I)| < \varepsilon$.

Obviously, every (P_0) integrable function is (P) integrable. The above definition comes from a condition in Bullen [1; p. 13] which he used to prove a more general Marcinkiewicz theorem. It is not the full continuity condition asked by Skvortsov, but it is the most we can achieve for the n-dimensional case using the present proof.

<u>**Lemma 1**</u>. *If f is (P) integrable to c on E, then it is (H) integrable to the same value c on E, i.e., for every $\varepsilon > 0$ there exists δ such that for any δ-fine division $D = \{(I, x)\}$ of E we have $|(D)\sum f(x)|I| - c| < \varepsilon$.*

The proof is identical to that of Theorem 8.7 [2; p.44]. Here (H) stands for Henstock.

A measurable function f defined on E is said to have LSRS (locally small Riemann sums) if for every $\varepsilon > 0$ there is a δ such that for any $t \in E$, any interval I_0 with $t \in I_0 \subset S(t, \delta(t)) \cap E$ and for any δ-fine division $D = \{(I, x)\}$ of I_0 we have $|(D)\sum f(x)|I|| < \varepsilon$.

<u>**Lemma 2**</u>. *If f is (H) integrable on E, then f has LSRS.*

The proof is identical to that of Theorem 17.2 [2; p.110].

<u>**Lemma 3**</u>. *If f is (H) integrable to c on E, then f is also (B) integrable to c on E, i.e., f is measurable and*

$$\inf_{\delta} \sup_{D} (D)\sum f(x)|I| = \sup_{\delta} \inf_{D} (D)\sum f(x)|I| = c,$$

where all the divisions D above are δ-fine divisions of E.

The proof follows from Lemma 2 and Theorem 17.9 [2; p.113]. Here (B) stands for Burkill. (The original Burkill integral uses interval functions and constant δ.)

<u>**Lemma 4**</u>. *If f is (B) integrable to c on E, then it is (P_0) integrable to c on E.*

Proof: Given δ, write for each interval $I_0 \subset E$

$$A(I_0) = \sup_{D} (D)\sum f(x)|I|, \; B(I_0) = \inf_{D} (D)\sum f(x)|I|,$$

where the supremum and the infimum are taken over all δ-fine divisions D of I_0. It is easy to see that A is δ-superadditive and B δ-subadditive. Let $x \in E$ and let I be an interval with $x \in I \subset S(x, \delta(x)) \cap E$. Then (I, x) is a δ-fine division of I whence $A(I) \geqq f(x)|I|$. It follows that $\underline{D}A(x) \geqq f(x)$. We see that A is a majorant of f. Similarly, B is a minorant of f. Now let $\varepsilon > 0$. Since f is (B) integrable, there is a δ such that $0 \leqq A(E) - B(E) < \varepsilon$. In view of the previous lemmas, f is also (H) integrable on E and has LSRS. It implies that both A and B satisfy the conditions required in the definition of the (P_0) integral. Hence f is (P_0) integrable on E.

Finally, combining the above lemmas we have proved the following

Theorem. *If f is (P) integrable to c on E, then f is (P_0) integrable to c on E.*

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