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DERIVATIVES AND CONVEXITY

Lemma 4.4 in [MW] says that if the composition of a function F strictly convex on an open interval containing the range of a derivative f is also a derivative, then both functions f and $F \circ f$ are Lebesgue functions. Theorem 4 of this note generalizes that result; f is there an n-tuple of derivatives and F is strictly convex on an open convex set containing the range of f. Theorems 5 and 8 deal with the one-dimensional case without the assumption that the domain of definition of F is open.

1. Notation. The symbols D, D^+, C_{ap}, L mean the systems of all derivatives, nonnegative derivatives, approximately continuous functions and Lebesgue functions on the interval $I = [0,1]$, respectively. Symbols like $\int_a^b f$ or $\int_S f$ mean the corresponding Lebesgue integrals. The letter n denotes a natural number and R^n the n-dimensional Euclidean space. For $z = (z_1, \dots, z_n) \in R^n$ we write $|z| = (z_1^2 + \cdots + z_n^2)^{1/2}$.

2. Lemma. *Let $f_1, \dots, f_n \in D$, $x \in I$. Set $f = (f_1, \dots, f_n)$, $b = f(x)$, $S = f(I)$. Let H be a function on S such that $H(b) = 0$, $H \circ f \in D$ and that for each $\varepsilon \in (0, \infty)$ we have*

$$\inf\{H(z)/|z-b|;\ z \in S,\ |z-b| > \varepsilon\} > 0. \tag{1}$$

Then

$$\frac{1}{y-x}\int_x^y |f-b| \to 0\ (y \to x, y \in I). \tag{2}$$

Proof. Let $\varepsilon \in (0,\infty)$ and let α be the infimum in (1). Set $\varphi = H \circ f$. Then $\varphi \geqq \alpha(|f-b| - \varepsilon)$ whence $|f-b| \leqq \alpha^{-1}\varphi + \varepsilon$ on I. Because $\varphi \in D$ and $\varphi(x) = 0$, we have $\limsup \frac{1}{y-x}\int_x^y |f-b| \leqq \varepsilon$ $(y \to x, y \in I)$ which proves (2).

3. Lemma. *Let $f, g \in C_{ap}$, $f \in D$ and $|g| \leqq f$. Then $g \in L$.* (See [M], 1.8.)

4. Theorem. *Let F be a strictly convex function on an open convex set G in R^n. Let $f_1, \dots, f_n \in D$, $f = (f_1, \dots, f_n)$. Suppose that $f(I) \subset G$ and that $F \circ f \in D$. Then $f_1, \dots, f_n$, $F \circ f \in L$.*

Proof. Let $b \in G$. It is well-known (see, e.g., [Mt], V, 1, Korollar 4) that there is a linear function λ such that $\lambda(b) = F(b)$ and $\lambda \leqq F$ on G. Since F is strictly convex, we have $\lambda < F$ on $G \setminus \{b\}$.

Now let ε be a positive number such that the set $A = \{z \in R^n;\ |z - b| = \varepsilon\}$ is a part of G. Set $H = F - \lambda$. Since F is continuous (see, e.g., [Mt], X, 1, Satz 2) and $H > 0$ on A, there is a $\beta \in (0, \infty)$ such that $H > \beta$ on A.

Let $z \in G$ and $|z - b| > \varepsilon$. Set $v = (z - b)/|z - b|$, $J = \{t \in (-\infty, \infty);\ b + tv \in G\}$, $h(t) = H(b + tv)$ $(t \in J)$. It is easy to see that h is (strictly) convex. Clearly $h(\varepsilon) > \beta$. Thus $h(t) \geqq th(\varepsilon)/\varepsilon > t\beta/\varepsilon$ for each $t \in J \cap (\varepsilon, \infty)$. It follows that $H(z) = h(|z-b|) > |z-b|\beta/\varepsilon$ so that, by 2, $f_1, \ldots, f_n \in L$. Thus $H \circ f \in C_{ap} \cap D^+$. By 3 we have $H \circ f \in L$ whence $F \circ f \in L$.

5. Theorem. *Let $f \in D$. Let F be a strictly convex function on $f(I)$ such that $F \circ f \in D$. Then $f \in L$. If, moreover, F is continuous, then also $F \circ f \in L$.*

Proof. Let $x \in I$, $b = f(x)$. Set $S = f(I)$. If $b \in \operatorname{int} S$, we get (2) as in the preceding proof. If, e.g., $b = \min S$, then (2) is obvious (since $f \geqq b$ and $f \in D$). Thus $f \in L$.

Now suppose that F is continuous. There is a linear function λ such that $F \geqq \lambda$ on S. Then $(F - \lambda) \circ f \in C_{ap} \cap D^+$ whence, by 3, $(F - \lambda) \circ f \in L$. Thus $F \circ f \in L$.

Remark. The example in 7 shows that the relation $F \circ f \in L$ may be false, if F is not continuous (even if f, $F \circ f \in D$ etc.).

We need first a lemma.

6. Lemma. *Let f be a nonnegative (Lebesgue) measurable function on I. Let $\frac{1}{x}\int_0^x f^2 \to 1$, $\lim \operatorname{ap} f(x) = 0$. Then $\frac{1}{x}\int_0^x f \to 0$ $(x \to 0+)$.*

Proof. Let $\varepsilon \in (0, \infty)$. Set $z_k = 2^{-k}$, $J_k = [z_k, 2z_k]$, $S_k = \{x \in J_k;\ f(x) > \varepsilon\}$ $(k = 1, 2, \ldots)$. Let β_k be the measure of S_k. Then $\frac{1}{z_k}\int_{J_k} f \leqq \frac{1}{z_k}\int_{S_k} f + \varepsilon$ and $\frac{1}{z_k}\int_{S_k} f \leqq (\frac{1}{z_k}\int_{J_k} f^2 \cdot \beta_k/z_k)^{\frac{1}{2}} \to 0$ $(k \to \infty)$ which easily implies our assertion.

7. Example. Let $F(z) = z^2 (z \in (0, \infty))$, $F(0) = 1$. Let f be a function such that $f \in C_{ap}$, f is positive and continuous on $(0, 1]$, $f(0) = 0$ and $\frac{1}{x}\int_0^x f^2 \to 1$ $(x \to 0+)$. (It is easy to construct such a function.) Then $F \circ f \in D \setminus C_{ap}$ and, by 6, $f \in D$.

8. Theorem. *Let $f \in D$. Suppose that f is not constant. Let F be a strictly convex function on $f(I)$ and let $F \circ f \in D \cap C_{ap}$. Then F is continuous.*

Proof. By 5 we have $f \in C_{ap}$. Define a function F_0 on $S = f(I)$ setting $F_0 = F$ on $\operatorname{int} S$ and $F_0(x) = \lim F(z)$ $(z \to x, z \in \operatorname{int} S)$ for $x \in S \setminus \operatorname{int} S$. Then F_0 is continuous on S so that $F_0 \circ f \in C_{ap}$. Set $\varphi = (F - F_0) \circ f$. Then $\varphi \in C_{ap}$. We see that φ is a Darboux function that has at most three values. Thus φ is constant, $\varphi = 0$ on I, $F = F_0$ on S, F is continuous.

Remark. Our proof of 8 would fail in more dimensions, because then the limit used there need not exist. (Take, e.g., $G = (0, \infty) \times (0, \infty) \cup (0,0)$, $F(x,y) = x^2/y + x^2 + y^2$ for $x, y > 0$, $F(0,0) = 0$.)

References

[M] Jan Mařík, *Transformation and multiplication of derivatives*, Contemporary Mathematics, Vol. 42 (1985), 119-134.

[Mt] Jürg T. Marti, *Konvexe Analysis*, Birkhäuser Verlag, 1977.

[MW] Jan Mařík and Clifford E. Weil, *Sums of powers of derivatives*, Proc. AMS, Vol. 112, No. 3, 807-817.

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