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R. Henstock, Department of Mathematics, University of Ulster, Coleraine, County Londonderry, Northern Ireland BT52 1SA

THE INTEGRAL OVER PRODUCT SPACES AND WIENER'S FORMULA

In Lebesgue integration with Wiener measure W over infinite dimensional Cartesian product spaces T of copies of the real line $\mathbf{R}$, and $T(N)$ the Cartesian product of N (finite) of the $\mathbf{R}$, let f be a function of $T(N)$ alone, and otherwise constant. For f Lebesgue integrable over T, f is Lebesgue integrable over $T(N)$ and Wiener's formula is

$$\int_T f dW = \int_{T(N)} f dW. \tag{1}$$

A few questions arise. Is (1) true for non-absolute integrals such as generalized Riemann integrals in division spaces? Can we generalize W? If the right-hand integral exists, does the left-hand integral exist? [4],**Theorem** 5, pp.223-224 (proof on p. 230) answers the first two questions positively, and the third if all divisions of T are the special kind given in the definition of the "Fubini property in common", [4],p.220 and [6],*Chapter* 5 ,p.149. However, even for $N = 1, T(N) = \mathbf{R}$, not all divisions of T are special and the question remains: if $T = T(N) \otimes T(-N), f : T(N) \to \mathbf{R}$ integrable over $T(N), g : T(-N) \to sing(1)$, is fg integrable over T? [5], **Theorem** 1, p.386, is relevant; for length in $\mathbf{R}$ as measure, and $T = \mathbf{R}^2, f, g$ Perron (gauge) integrable over intervals of $\mathbf{R}, fg$ is gauge integrable over the Cartesian product of the intervals. [3],**Theorem 11**, p. 83, is a more general result for a general product space and products of interval-point $VB*$ functions $h_j (j = 1, 2)$, with a suggested extension to $VBG*$ functions. $h_2(I_2, t_2)$ finitely additive gives (1). As in [3], [4], [6], *sections* 2.7, 2.8, 5.1, definitions are as follows.

In the space T of points we choose a family $\mathcal{T}$ of some non-empty subsets called *(generalized) intervals* I, a fixed non-empty family $\mathcal{U}^1$ of interval-point pairs $(I, t)(I \in \mathcal{T}, t \in T)$, and (controlling integration) non-empty families $\boldsymbol{A}$ of some non-empty subsets $\mathcal{U} \subseteq \mathcal{U}^1$.

An elementary set E is an interval or a union of a finite number of mutually disjoint intervals. A subset $\mathcal{U} \subseteq \mathcal{U}^1$ *divides* E if, for a *finite* subset $\mathcal{E} \subseteq \mathcal{U}$, called a *division of* E *from* $\mathcal{U}$, the $(I,t) \in \mathcal{E}$ have mutually disjoint I(called *partial intervals of* E, and a *partition of* E, *from* $\mathcal{U}$) with union E. A non-empty $\mathcal{P} \subseteq \mathcal{E}$ is a *partial division of* E *from* $\mathcal{U}$, the union of I from $(I,t) \in \mathcal{P}$ is a *partial set* P *of* E *that comes from* $\mathcal{E}$ *and* $\mathcal{U}$, and P *is proper* if $P \neq E$. For $\mathcal{U}.E$ the set of all $(I,t) \in \mathcal{U} \subseteq \mathcal{U}^1$ with I a partial interval of E,let $\boldsymbol{A}|E$ be the set of all $\mathcal{U}.E$ dividing E with $\mathcal{U} \in \boldsymbol{A}$, and $E * .\mathcal{U}$ the set of all t with $(I,t) \in \mathcal{U}.E$ for some I. The *star set* $E*$ is the intersection of $E * .\mathcal{U}$ for all $\mathcal{U} \in \boldsymbol{A}|E$.

Taking $\boldsymbol{A}|E$ non-empty (saying that $\boldsymbol{A}$ *divides* E), we need $\boldsymbol{A}$ *directed for divisions of* E (*i.e.* given $\mathcal{U}_j \in \boldsymbol{A}|E(j = 1,2)$, a $\mathcal{U} \subseteq \mathcal{U}_1 \cap \mathcal{U}_2$ is in $\boldsymbol{A}|E$). This is the *direction as* $\mathcal{U}$ *shrinks. A restriction of* $\mathcal{U}$ *to a partial set* P is a non-empty family $\mathcal{U}_1 \subseteq \mathcal{U}.P$. We assume that $\boldsymbol{A}$ *has the restriction property* (*i.e.* for each elementary set E, each partial set P, and each $\mathcal{U} \in \boldsymbol{A}|E$, there is in $\boldsymbol{A}|P$ a restriction of $\mathcal{U}$ to P) and that $\boldsymbol{A}$ *is additive* (*i.e.* given disjoint elementary sets E_j and $\mathcal{U}_j \in \boldsymbol{A}|E_j(j = 1,2)$, a $\mathcal{U} \subseteq \mathcal{U}_1 \cup \mathcal{U}_2$ is in $\boldsymbol{A}|E_1 \cup E_2$). Such a $(T,\mathcal{T},\boldsymbol{A})$ is an *additive division space* in [6] (previously called a *division space.*) We integrate functions $h : \mathcal{U}^1 \to \mathbf{R}$ or $\mathcal{C}$, *e.g.* $h(I,t) = f(t)m(I)$ for a measure m; a number $H(E)$ is the $\boldsymbol{A}$-*integral of* h *over the elementary set* E if, given $\varepsilon > 0$, a $\mathcal{U} \in \boldsymbol{A}|E$ has every division $\mathcal{E}$ of E from $\mathcal{U}$ satisfying

$$|(\mathcal{E})\sum h - H(E)| < \varepsilon,$$

$(\mathcal{E})\sum$ denoting summation over the $(I,t) \in \mathcal{E}$. In an additive division space an $h : \mathcal{U}^1 \to \mathcal{C}$, $\boldsymbol{A}$-integrable over E, is $\boldsymbol{A}$-integrable over every partial set P of E, say to $H(P)$, *finitely additive in* P (*i.e.* if $P_j(j = 1,2)$ are disjoint partial sets of E then

$$H(P_1) + H(P_2) = H(P_1 \cup P_2),$$

and, given $\varepsilon > 0$, a $\mathcal{U} \in \boldsymbol{A}|E$ is such that for every division $\mathcal{E}$ of E from $\mathcal{U}$,

$$(\mathcal{E})\sum |h(I,t) - H(I)| < \varepsilon. \tag{2}$$

See [6],**Theorem** 2.5.2, p. 84;**Theorem** 2.5.5 (2.5.15), p. 87; without additivity of $(T,\mathcal{T},\boldsymbol{A})$. More generally, for $V(h;\mathcal{U};E)$ the supremum for all divisions $\mathcal{E}$ of E from $\mathcal{U} \in \boldsymbol{A}|E$, of

$$(\mathcal{E})\sum |h(I,t)|,$$

the variation $V(h;\boldsymbol{A};E)$ *of* h *over* E, is, as $\mathcal{U}$ shrinks,

$$\inf[V(h;\mathcal{U};E):\mathcal{U}\in\boldsymbol{A}|E]=\limsup(\mathcal{E})\sum|h(I,t)|.$$

In (2), $V(h-H;\boldsymbol{A};E)=0$ ($h-H$ *has variation zero*). If $X\subseteq T$ we write $V(h;\mathcal{U};E;X)$ and $V(h;\boldsymbol{A};E;X)$ for $V(h\chi(X;.);\mathcal{U};E)$ and $V(h\chi(X;.);\boldsymbol{A};E)$, respectively, $\chi(X;t)$ being the characteristic function or indicator of X. *A majorant of* $h:\mathcal{U}^1\to\mathbf{R}$ is an $M:\mathcal{T}\to\mathbf{R}$ such that for some $\mathcal{U}\in\boldsymbol{A}|E$, $h(I,t)\le M(I)((I,t)\in\mathcal{U})$. An $h:\mathcal{U}^1\to\mathcal{C}$ *is ultimately finitely additive in* E if, for some $\mathcal{U}\in\boldsymbol{A}|E, h$ is finitely additive in $\mathcal{U}$, independent of t. If the majorant $J(I)$ is the finite supremum of $(\mathcal{E})\sum h$ for all divisions $\mathcal{E}$ of I from some $\mathcal{U}\in\boldsymbol{A}|I$, then J *is finitely superadditive* (i.e. $(\mathcal{E})\sum J\le J(I)$ for all divisions $\mathcal{E}$ of I). However, at times we need ultimately finitely additive majorants M satisfying

$$M(E)\le KJ(E) \tag{3}$$

for some fixed number $K\ge 1$. No problem occurs if $T=\mathbf{R}$. Writing $J(u,v)$ for $J([u,v))$,

$$J(a,u)+J(u,v)\le J(a,v), J(u,v)\le J(a,v)-J(a,u)(a<u<v\le b)$$

and in $[a,b)$ we take $K=1$ and $J(a,v)-J(a,u)$ for $M(u,v)$. [1] shows difficulties in $\mathbf{R}^2$.

In special cases more can be proved. J.Mařik gave me an honours project by L.Trudzik that translates and discusses [7], including a construction of a finitely additive majorant of $f(x)G(I)$ where $fG, |f|G$ are Perron integrable and G is non-negative and finitely additive, so that fG is Radon (Lebesgue-Stieltjes) integrable. [1] gives marginally better results; W. F. Pfeffer says that even after 25 years it has not yet been superseded, so that here is another task.

Continuing with the definitions, $(T,\mathcal{T},\boldsymbol{A})$ *is fully decomposable* (respectively, *decomposable*, or *measurably decomposable* relative to a measure or measure space) if to every family (respectively, countable family or countable family of measurable sets) $\mathcal{X}$ of mutually disjoint subsets $X\subseteq T$ and every $\mathcal{U}(.):\mathcal{X}\to\boldsymbol{A}|E$, there is a $\mathcal{U}\in\boldsymbol{A}|E$ with $\mathcal{U}[X]\equiv\{(I,t):(I,t)\in\mathcal{U},t\in X\}\subseteq\mathcal{U}(X)(X\in\mathcal{X})$. If $\mathcal{U}[X]=\mathcal{U}(X)[X](X\in\mathcal{X}),\mathcal{U}$ is the *diagonal* of the $(\mathcal{U}(X),\mathcal{X})$. A fully decomposable additive division space is *stable* (*i.e.* for a

$\mathcal{U}(E) \in \boldsymbol{A}|E$, every $\mathcal{U}_1 \in \boldsymbol{A}|E$ with $\mathcal{U}_1 \subseteq \mathcal{U}(E)$ has $E*.\mathcal{U}_1 = E*.\mathcal{U}(E) = E*$). See [6], p. 43, before (1.1.2).

A Cartesian product $(T_z, \mathcal{T}_z, \boldsymbol{A}_z)$ of additive division spaces $(T_u, \mathcal{T}_u, \boldsymbol{A}_u)$ $(u = x, y)$, possibly similar, possibly very different, has $T_z = T_x \otimes T_y, \mathcal{T}_z$ the family of $I_x \otimes I_y$ for all $I_u \in \mathcal{T}_u(u = x, y)$, and $\boldsymbol{A}_z$ based on them and the family $\mathcal{U}_z^1$ of $(I_x \otimes I_y, (x, y))$, written $(I_x, x) \otimes (I_y, y)$, for all $(I_u, u) \in \mathcal{U}_u^1(u = x, y)$. We suppose that *$\boldsymbol{A}_x, \boldsymbol{A}_y, \boldsymbol{A}_z$ have the Fubini property in common* (really two properties). First, for E_u an arbitrary elementary set in $T_u(u = x, y)$, $E_z = E_x \otimes E_y$, and arbitrary $\mathcal{U}_z \in \boldsymbol{A}_z|E_z$, there is a $\mathcal{U}_y(.) : E_x* \to \boldsymbol{A}_y|E_y$, and to each collection of divisions $\mathcal{E}_y(x)$ of E_y from $\mathcal{U}_y(x)$, one division for each such x, there is a $\mathcal{U}_x \in \boldsymbol{A}_x|E_x$ such that $(I_x, x) \otimes (I_y, y) \in \mathcal{U}_z$ when $(I_x, x) \in \mathcal{U}_x, (I_y, y) \in \mathcal{E}_y(x)$. Secondly, interchange x, y, but with x first in all Cartesian products.

Note that star sets give location, particularly in a stable division space.

For a particular construction let $\mathcal{U}_u(.) : T_z \to \boldsymbol{A}_u|E_u(u = x, y)$. Let $\mathcal{U}_z$ be the family of all $(I_x, x) \otimes (I_y, y)$ with $(I_u, u) \in \mathcal{U}_u(z)(u = x, y; z = (x, y))$. $\boldsymbol{A}_z$ is the family of finite unions of $\mathcal{U}_z$ for all finite unions of disjoint products $E_x \otimes E_y$. Calling this the *product division space*, by [6], **Theorem** 5.1.1, p. 149, and fully decomposable division spaces, the $\boldsymbol{A}_x, \boldsymbol{A}_y, \boldsymbol{A}_z$ have the Fubini property in common. (4) seems to follow,

(4) If $I_x \otimes I_y$ is a partial interval of $E_x \otimes E_y$, there are elementary

sets E_{1u} disjoint from I_u with $I_u \cup E_{1u} = E_u(u = x, y)$.

We can call the integral from the product division space a *product space integral.*

Necessary conditions in **Theorem 1** are sufficient in special cases (**Theorem 2**).

Theorem 1. *Let $(T_u, \mathcal{T}_u, \boldsymbol{A}_u)(u = x, y)$ be fully decomposable and $\boldsymbol{A}_x, \boldsymbol{A}_y, \boldsymbol{A}_z$ have the Fubini property in common. Let E_x, E_y be elementary sets with $E_z = E_x \otimes E_y$. Let*

$$h_x(I_x, x) h_y(x; I_y, y) \tag{5}$$

be $\boldsymbol{A}_z$-integrable over E_z with X the set of x for which $h_y(x; .)$ is ultimately finitely additive. Then h_x is VBG in $E_x * \setminus X$.*

Let

$$(6) \qquad h_y(I_y, y)h_x(y; I_x, x)$$

be $\boldsymbol{A}_z$-integrable over E_z with Y the set of y for which $h_x(y;.)$ is ultimately finitely additive. Then h_y is VBG in $E_y * \backslash Y$.*

$$(7) \qquad f(x,y)h_x(I_x,x)h_y(I_y,y)$$

has the form of (5), (6); and if the products there are equal, they have the form (7).

Proof. In (5) $h_y(x;.)$ is $\boldsymbol{A}_y$-integrable over E_y to (say) $H_y(x; E_y)$ for h_x-almost all $x \in E_x*$ (the unsymmetrical Fubini theorem, [6], **Theorem** 5.1.2, p. 150.) As the exceptional set X_o has h_x-variation 0 it is included in $E_x * \backslash X$ in (5). For all $x \in E_x * \backslash(X \cup X_o)$ each $\mathcal{U}_y(x) \in \boldsymbol{A}_y|E_y$ contains a division $\mathcal{E}_y(x)$ of E_y with

$$(8) \qquad g(x) \equiv (\mathcal{E}_y(x)) \sum |h_y(x; I_y, y) - H_y(x; I_y)| > 0,$$

or else $h_y(x;.)$ would ultimately be $H_y(x;.)$, finitely additive. By the Fubini property, for $\varepsilon > 0$ there are $\mathcal{U}_z, \mathcal{U}_y(x), \mathcal{E}_y(x), \mathcal{U}_x$ with the given properties, so that

$$(9) \qquad (\mathcal{E}_x) \sum |h_x(I_x,x)| g(x) < \varepsilon, g(x) > 0 (x \in E_x * \backslash(X \cup X_o).)$$

As $\mathcal{E}_x$ is an arbitrary division of E_x from $\mathcal{U}_x$ we take the sets of x where, respectively, $1/n > g(x) \geq 1/(n+1)$, $(n = 0,1,2,...)$. By (9), h_x is VB* in each, and so VBG* in $E_x * \backslash X$. For (6) interchange x and y.

The converse of **Theorem 1** would at least involve proof of the integrability of (7) over the product set. But a Sierpiński [8] construction, a non-measurable plane set meeting every line parallel to the x and y axes in at most two points, shows that a converse of **Theorem 1** can only be partial, with no easy proof except in simple cases.

Theorem 2. *For the product division space $(T_z, \mathcal{T}_z, \boldsymbol{A}_z)$ with* (4),*of $(T_u, \mathcal{T}_u, \boldsymbol{A}_u)$, $(u = x, y)$, fully decomposable additive division spaces, let h_u be $\boldsymbol{A}_u$-integrable to $H_u(P_u)$ over the partial sets P_u of an elementary set E_u and let M_u be an ultimately finitely additive majorant of $|h_u - H_u|$ satisfying* (3) *for some $K_u \geq 1$ in $E_u (u = x,y)$. Then $h_x h_y$ is $\boldsymbol{A}_z$-integrable to $H_x(E_x)H_y(E_y)$ over $E_z \equiv E_x \otimes E_y$ in the following cases :*

1. *if h_u is ultimately finitely additive over E_u, VBG* or not* $(u = x, y)$;

2. *if h_u is VBG* and ultimately finitely additive over E_u*,but h_v is not ultimately finitely additive over $E_v *$* $(u = x$ *and* $v = y$, or $u = y$ *and* $v = x)$;

3. *if h_u is VBG* and not ultimately finitely additive over $E_u *$* $(u = x, y)$.

Proof. In (10) let $\mathcal{E}_z$ be a division of E_z from a $\mathcal{U}_z$ for which h_x, h_y are finitely additive, and let $(I_x, x) \otimes (I_y, y) \in \mathcal{E}_z$. By (4) there is a partition of E_x that includes I_x. Repeating for each interval-point pair in $\mathcal{E}_z$ and using [6], **Theorem** 2.7.1, p. 93 (that needs the additivity of the space), with direction in $\boldsymbol{A}_x$, we have a partition of E_x that partitions every I_x with $(I_x, x) \otimes (I_y, y) \in \mathcal{E}_z$. The corresponding partition $\mathcal{P}_z$ of E_z is a collection of partitioned strips $I_{ox} \otimes E_y$. By finite additivity of h_x the sum of $h_x h_y$ over $\mathcal{E}_z$ is equal to the sum over $\mathcal{P}_z$, which is $H_x(E_x)H_y(E_y)$.

For (11) we need only take $u = x, v = y, h_x = H_x$, finitely additive, and VB* in each of a sequence (X_n) of mutually disjoint sets with union T_x, proving that

$$h_x h_y - H_x H_y = H_x(h_y - H_y) \tag{10}$$

has variation zero. We temporarily omit suffices x. As $V(H; \boldsymbol{A}; E; X)$ is an outer measure in X ([6],**Theorem** 2.2.1 (2.2.1), p. 71),

$$\begin{gathered} V_n \equiv V(H; \boldsymbol{A}; E; X_n) = 0 \text{ (all } n\text{) imply} \\ V(H; \boldsymbol{A}; E) = V(H; \boldsymbol{A}; E; T) = 0. \end{gathered} \tag{11}$$

H is finitely additive, hence $H = 0$ and (13) is 0. Thus we forget (14), taking m the smallest integer with $V_m > 0$, aggregating with X_m all X_n with $V_n = 0$, again using the outer measure property. Thus we take $V_n > 0$ (all n) and again use suffices x.

In the product division space construction, $(T_y, \mathcal{T}_y, \boldsymbol{A}_y)$ being fully decomposable, we replace $\mathcal{U}_y(z)$ by $\mathcal{U}_y(x)$, chosen so that in (2) all divisions $\mathcal{E}_y(x)$ over E_y from $\mathcal{U}_y(x)$ satisfy

$$(E_y(x)) \sum |h_y(I_y, y) - H_y(I_y)| < \varepsilon . 2^{-n-1} V_n^{-1} K_y^{-1} (x \in X_n). \tag{12}$$

From (15) and (3)($K = K_y$) a finitely additive majorant M_{ny} of $|h_y - H_y|$ exists with

$$(13) \qquad M_{ny}(E_y) < \varepsilon.2^{-n-1}V_n^{-1}.$$

A $\mathcal{U}_x$ and a finitely superadditive majorant s_{nx} of $|H_x|\chi(X_n;.)$ using $\mathcal{U}_x$, exist with

$$(14) \quad s_{nx}(E_x) \leq V(H_x;\mathcal{U}_x;E_x;X_n) \leq 2V(H_x;\boldsymbol{A}_x;E_x;X_n) = 2V_n.$$

Using (16),(17), and the proof of (10)(interchanging x and y),

$$|H_x(I_x)||h_y(I_y,y) - H_y(I_y)| \leq s_{nx}(I_x)M_{ny}(I_y) \leq \sum_{m=1}^{\infty} s_{mx}(I_x)M_{my}(I_y)$$

($x \in X_n$),

$$(\mathcal{E}_z)\sum|H_x(I_x)||h_y(I_y,y) - H_y(I_y)| \leq \sum_{m=1}^{\infty} s_{mx}(E_x)M_{my}(E_y) < \varepsilon.$$

Hence (11). For (12), if, because of (3), there is a suitable finitely additive majorant M_u of $|h_u - H_u|(u = x, y)$, then,using the proof of (10),

$$(15) \; (\mathcal{E}_z)\sum|h_x - H_x||h_y - H_y| \leq (\mathcal{E}_z)\sum M_xM_y \leq M_x(E_x)M_y(E_y),$$

suitably small, so that $(h_x - H_x)(h_y - H_y)$ has variation zero. As

$$h_xh_y - H_xH_y = (h_x - H_x)(h_y - H_y) + H_x(h_y - H_y) + (h_x - H_x)H_y,$$

(18),(11), and the given conditions show that h_xh_y is product space integrable to H_xH_y.

By Mařík [7] the existence of the ultimately finitely additive majorant M occurs when $h = f \triangle G$ and $|h|$ are gauge integrable with $\Delta G \geq 0$. For the generalization of Wiener's formula (1) in which h_y is ultimately finitely additive, if h_xh_y is $\boldsymbol{A}_z$-integrable, the Fubini property and **Theorem 1** show that h_x is ultimately finitely additive, or $\boldsymbol{A}_x$-integrable with h_y VBG*. Conversely, by **Theorem 2**, if h_y is ultimately finitely additive, say to H_y, with h_x $\boldsymbol{A}_x$-integrable to H_x and either ultimately finitely additive or such that $|h_x - H_x|$ has an ultimately finitely additive majorant satisfying (3) with h_y VBG*, then h_xh_y is product space integrable to H_xH_y. These results apply even when h_x, h_y are not necessarily non-negative, and particularly in Feynman integration. Also this paper shows that in [2],p.330, the VBG* condition is necessary even though it is not used in the proof of [2], **Theorem** 4.

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