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James Foran, Department of Mathematics, University of Missouri-Kansas City
Kansas City, MO 64110

SEQUENCE CONDITIONS WHICH IMPLY APPROXIMATE CONTINUITY

A function is approximately continuous at a point x if, on removal of a set E which has a density 0 at x, the function is continuous at x with respect to E^c. This suggests that, in some probability sense, it is unlikely that a sequence approaching the point x should contain infinitely many points of the set E.

The purpose of this paper is to examine some probability spaces, whose elements are sequences tending to a point x, for which a function, continuous at the point x with respect to a collection of sequences whose probability is 1, is necessarily approximately continuous at x.

The following conventions, standard definitions and notation will be needed. The Lebesgue measure of a subset E of the line will be denoted by $m(E)$. All sets under consideration will be Lebesgue measurable and all functions will be measurable functions; the integral used in this paper will be the Lebesgue integral. The <u>characteristic function</u> of a set E, denoted by C_E, satisfies $C_E(x) = 1$ if $x \in E$, $C_E(x) = 0$ if $x \notin E$. The <u>density</u> of a set E at x, written $D_E(x)$ is $\lim m(E \cap I)/m(I)$ provided the limit exists. Here the limit is taken over intervals I with x in I and $m(I)$ approaching 0. The <u>upper density</u> of E at x, written $\bar{D}_E(x)$, is $\overline{\lim}\ m(E \cap I)/m(I)$; the <u>lower density</u> of E at x, written $\underline{D}_E(x)$, is $\underline{\lim}\ m(E \cap I)/m(I)$. A point x is a <u>point of density</u> of E if $D_E(x) = 1$; a point x is a <u>point of dispersion</u> of E if $D_E(x) = 0$. A function f is <u>approximately continuous</u> at x if x is a point of density of a set E and f is continuous at x with respect to E. The collection of measurable sets E for which each point of E is a point of density of E form a topology called the <u>density topology</u>.

Without loss of generality, we will be concerned with approximate continuity of a function at the point 0; we will also only consider approximate continuity at 0 from the right and frequently use only the right hand density at 0, $\lim_{h \to 0^+} m(E \cap (0,h))/h$, which will be written $D^r_E(0)$ and the corresponding upper and lower densities $\bar{D}^r_E(0)$ and $\underline{D}^r_E(0)$. Clearly, a set E has 0 as a point of density (resp., dispersion) iff the value of both the right and left hand densities at 0 is 1 (resp., 0).

Consider the following simple example of a probability space X whose elements are sequences decreasing to 0: fix a sequence $\{t_n\}$ decreasing to 0 and let X consist of all sequences of the form $\{xt_n\}$ for x in $(0,1)$; let the probability of a collection C of sequences from X corresponding to a measurable subset of $(0,1)$ be equal to $m(\{x \in (0,1) : \{x \cdot t_n\} \in C\})$.

We begin by characterizing the sequences $\{t_n\}$ for which the following property holds:

(*) Whenever $f(xt_n)$ approaches $f(0)$ for almost every x in $(0,1)$, then f is approximately continuous at 0 from the right.

Theorem 1. *Let $\{t_n\}$ be a sequence which is decreasing to 0. In order that (*) hold for each measurable function f, it is necessary and sufficient that there be an $r > 0$ so that for each n, $t_{n+1} > rt_n$ or, what amounts to the same thing, that $\underline{\lim}\, t_{n+1}/t_n > 0$.*

Proof. Suppose $\{t_n\}$ decreasing to 0 is given. Consider any measurable function f. Suppose $t_{n+1} > rt_n$ for some $r > 0$ and that $f(xt_n)$ approaches $f(0)$ for almost every x in $(0,1)$. By selecting, if necessary, a subsequence of $\{t_n\}$, we may presume for each n, that $t_{n+1} \leq t_n/2$ and $t_{n+1} \geq rt_n$ for some $r > 0$. To see that f is approximately continuous at 0 from the right, let

$$A_{N,k} = \{x : n > N \text{ implies } |f(xt_n) - f(0)| < 1/k\}.$$

Then for every k there is N_k so that $m(A_{N_k,k}) > 1 - 1/k$. Let

$$B_k = \{x : xt_n^{-1} \in A_{N_k,k} \text{ for some } n \text{ with } N_k \leq n < N_{k+1}\}.$$

Let $E = \cup B_k \cap (t_{N_{k+1}}, t_{N_k})$. If $x \in E$ and $t_{N_{k+1}} < x < t_{N_k}$, then $x \in B_k$ and $xt_n^{-1} \in A_{N_k,k}$ for some n with $N_k \leq n < N_{k+1}$. Thus $f(x) = f(xt_n^{-1}t_n)$ and $|f(x) - f(0)| < 1/k$. Thus f is continuous on the right at 0 with respect to E and it remains to show that E has 0 as a point of density on the right. To see this, suppose $h > 0$ is given with $t_{n+1} \leq h \leq t_n$ where $N_k \leq n < N_{k+1}$. Since $m(A_{N_k,k}) > 1 - 1/k$, one has $m(\{x \in (0,t_n) : xt_n^{-1} \in A_{N_k,k}\}) > (1 - 1/k)t_n$. Because $t_n - t_{n+1} > t_n/2$ and thus $(t_n - t_{n+1}) \cdot 2/k \geq t_n/k$, it follows that

$$\begin{aligned} m(E \cap (t_{n+1}, t_n)) &\geq (1 - 1/k)t_n - t_{n+1} = t_n - t_{n+1} - t_n/k \\ &\geq (1 - 2/k)(t_n - t_{n+1}). \end{aligned}$$

Then $m(E \cap (0,h))/h$ is at least as large as the quantity obtained by assuming that $E \cap (t_{n+1}, h) = \phi$ and $h = t_{n+1} + (2/k)(t_n - t_{n+1})$. That is

$$\frac{m(E \cap (0,h))}{h} \geq \frac{(1 - 1/k)t_{n+1}}{t_{n+1} + (2/k)(t_n - t_{n+1})} \geq \frac{1 - 1/k}{1 + (2/k)(1 - r)/r}.$$

Here, the last inequality is due to the fact that $t_{n+1} \geq rt_n$ and thus $t_n - t_{n+1} \leq (1-r)t_n \leq t_{n+1}(1-r)/r$. Since, as h approaches 0, k approaches ∞, $m(E \cap (0,h))/h$ approaches 1 as h approaches 0. Thus E has 0 as a point of density on the right. Now, to see the converse, suppose $\{t_n\}$ decreases to 0 and $\underline{\lim}\ t_{n+1}/t_n = 0$. We must construct a measurable function f which is not approximately continuous at 0 from the right and yet for almost every x in $(0,1)$, $f(xt_n)$ approaches $f(0)$. Since $\underline{\lim}\ t_{n+1}/t_n = 0$, there is a subsequence $\{n_k\}$ of the natural numbers so that $t_{n_k+1} < k^{-1} \cdot 2^{-k} \cdot t_{n_k}$. The sequence $\{n_k\}$ can also be chosen so that $t_{n_{k+1}} < 2^{-k} t_{n_k+1}$. Let $E = \bigcup_k (t_{n_k+1}, k \cdot t_{n_k+1})$ and $f(x) = C_E(x)$. Then f is not approximately continuous at 0 because $f(0) = 0$, $f(x) = 1$ for x in E and, indeed, $\bar{D}^r_E(0) = 1$. However,

$$\begin{aligned}
\lim_{N\to\infty} m(\{x : f(xt_n) &= 1 \text{ for some } n > N\}) \\
&\leq \lim_{K\to\infty} \sum_{k=K}^{\infty} t_{n_k}^{-1}(k \cdot t_{n_k+1} + (k+1) \cdot t_{n_{k+1}+1} + \ldots) \\
&\leq \lim_{K\to\infty} \sum_{k=K}^{\infty} t_{n_k}^{-1} \cdot 2^{-k} t_{n_k} \\
&\leq \lim_{K\to\infty} \sum_{k=K}^{\infty} 2^{-k} = 0.
\end{aligned}$$

Thus, for almost every x in $(0,1)$ there is an N so that $f(xt_n) = 0$ when $n > N$. Consequently, $f(xt_n)$ approaches $0 = f(0)$ for almost every x in $(0,1)$.

We now characterize the measurable functions f for which there is some $\{t_n\}$ decreasing to 0 so that $f(xt_n)$ approaches $f(0)$ for almost every x in $(0,1)$.

Theorem 2. *Let f be a measurable function defined on a neighborhood of 0. There is a set $E \subset (0,1)$ so that $\underline{D}^r_E(0) = 0$ and f is continuous on the right at 0 with respect to E^c iff there is a sequence $\{t_n\}$ decreasing to 0 so that $\lim_n f(xt_n) = f(0)$ for almost every x in $(0,1)$.*

Proof. Suppose there is E with $\underline{D}^r_E(0) = 0$ and f is continuous on the right at 0 with respect to E^c. Choose t_n so that $m(E \cap (0,t_n)) < 2^{-n} t_n$. For $c \geq 0$, let $c \cdot A = \{cx : x \in A\}$ and note that $m(cA) = c \cdot m(A)$. Then

$$\begin{aligned}
&m(\{x \in (0,1) : f(xt_n) \text{ does not approach } f(0)\}) \\
&\leq \sum_{n=N}^{\infty} m(\{x \in (0,1) : xt_n \in E\}) = \sum_{n=N}^{\infty} m(t_n^{-1} \cdot (E \cap (0,t_n))) \\
&\leq \sum_{n=N}^{\infty} t_n^{-1}\, m(E \cap (0,t_n)) \leq \sum_{n=N}^{\infty} 2^{-n} = 2^{-N+1}.
\end{aligned}$$

Since this is true for each N, $f(xt_n)$ approaches $f(0)$ for almost every x in $(0,1)$. For the converse, suppose f is measurable and there is a sequence $\{t_n\}$ decreasing to 0 so that $\lim_n f(xt_n) = f(0)$ for almost every x in $(0,1)$. Choose N_1 so that $t_{N_1} < 1/2$. Then there is $N_1' > N_1$ so that

$$E_1 = \{x \in (t_{N_1'}, 1) : |f(xt_n) - f(0)| < 1 \text{ when } n > N_1'\}$$

has $m(E_1 \cap (t_{N_1}, 1)) > 1/2$. Then

$$m(\bigcup_{n \geq N_1'} t_n \cdot (E_1 \cap (0, t_{N_1'}))) > 1/2 t_{N_1'}$$

and there is $N_1'' > N_1'$ so that

$$m(\bigcup_{N_1' \leq n \leq N_1''} t_n \cdot (E_1 \cap (t_{N_1''}, t_{N_1'}))) > 1/2(t_{N_1'}).$$

In general, choose the number N_k so that $N_k > N_{k-1}''$ and $t_{N_k} < 1/(k+1)$. then there is N_k' so that

$$E_k = \{x \in (t_{N_k'}, 1) : |f(xt_n) - f(0)| < 1/k \text{ when } n > N_k'\}$$

has $m(E_k \cap (t_{N_k'}, 1)) > k/(k+1)$. Then

$$m(\bigcup_{n \geq N_k'} t_n \cdot (E_k \cap (0, t_{N_k'}))) > k/(k+1) \cdot t_{N_k'}$$

and there is $N_k'' > N_k'$ so that

$$m(\bigcup_{N_k' \leq n \leq N_k''} t_n \cdot (E_k \cap (t_{N_k''}, t_{N_k'}))) > k/(k+1) \cdot t_{N_k'}.$$

If

$$E^c = \bigcup_{k=1}^{\infty} \bigcup_{N_k' \leq n \leq N_k''} t_n \cdot (E_k \cap (t_{N_k''}, t_{N_k'})),$$

then the upper density of E^c at 0 from the right is 1 because it is greater than each number $k/k+1$. Thus $\underline{D}_E^r(0) = 0$. Also f is continuous at 0 with respect to the points of E^c because $|f(x) - f(0)|$ is less than $1/k$ when x belongs to

$$\bigcup_{N_k' \leq n \leq N_k''} t_n \cdot (E_k \cap (t_{N_k''}, t_{N_k'})).$$

We now turn to consider some probability spaces whose elements are countable subsets of $(0,\infty)$. We will first construct a space to examine density at 0 by 'turning around' a frequently used probability space whose elements can be considered to be increasing sequences $\{t_n\}$ with $\lim t_n = \infty$; namely, a Poisson process. (See, for example, [2] for the development of such processes.) The standard Poisson process describes 'arrival times' and involves a 'rate of arrival', $c > 0$. It is a probability space whose elements are increasing sequences S where the probability, $(c)Pr$, is determined by the formulas:

$(c)Pr\{S : \#(S \cap (a,b)) = n\}) = e^{-ct}(ct)^n/n!$
where $t = b - a$, and for disjoint intervals
(a_i, b_i) and non-negative integers n_i,
$(c)Pr(\{S : \#(S \cap (a_i, b_i)) = n_i\}) =$
$\Pi e^{-ct_i}(ct_i)^{n_i}/n_i!$ where $t_i = b_i - a_i$. (Here
$\#(x \cap y)$ denotes the cardinality of $x \cap y$.)

In particular, for measurable sets $A \subset (0, \infty)$,

$$(c)Pr(\{S : \#(S \cap A) < \infty\}) = 1 \text{ iff } m(A) < \infty.$$

For $c = 1$ fixed and $S = \{t_n\}$, let $1/S$ denote $\{1/t_n\}$. If A is a collection of sequences S increasing to ∞, and $1/A$ denotes the collection of sequences $1/S$ for S in A, put $Pr'(1/A) = (c)Pr(A) = Pr(A)$.

We are only interested in a simple situation: if f is approximately continuous at 0 on the right, there is a set E with $D_E^r(0) = 0$ so that if t_n approaches 0 and only finitely many $t_n \in E$, then $\lim f(t_n) = f(0)$; thus, given a set E with $D_E^r(0) = 0$, we want to know whether $Pr'(\{1/S : \#(1/S \cap E^{-1}) < \infty\}) = 1$. But this is equivalent to $Pr(\{S : \#(S \cap E^{-1}) < \infty\}) = 1$ where $E^{-1} = \{x^{-1} : x \in E\}$. As noted above, this probability is 1 iff $m(E^{-1}) < \infty$.

Since, for an interval $(a,b) \subset (0,1)$, $m((a,b)^{-1}) = a^{-1} - b^{-1} = \int_a^b t^{-2}dt$, it follows that, for a measurable set $E \in (0,1)$, $m(E^{-1}) = \int_E t^{-2}dt$. Thus $Pr'(1/S : \#(1/S \cap E) < \infty) = 1$ iff $\int_E 1/t^2 dt < \infty$. By considering an open set $G = \cup(a_i, b_i)$ containing E,

$$\int_G t^{-2}dt < \infty \text{ iff } \Sigma \int_{a_n}^{b_n} t^{-2}dt < \infty \text{ iff } \Sigma(1/a_n - 1/b_n) = \Sigma \frac{b_n - a_n}{a_n b_n} < \infty.$$

This condition implies $D_E^r(0) = 0$ because it is equivalent to $\lim_{h\to 0} \Sigma_h \frac{b_n - a_n}{a_n b_n} = 0$ where Σ_h is the sum over all n with $a_n < h$; $D_E^r(0) = 0$ is equivalent to the existence of an open set $G = \cup(a_i, b_i)$ so that $\lim \Sigma_h \frac{b_n - a_n}{b_N} = 0$ where $h \in [a_N, b_N]$ defines N and again Σ_h is the sum over all n with $a_n < h$. Note that this implies that if $m(E^{-1}) < \infty$, then $D_E^r(0) = 0$. The converse is far from true: for example, if $E^{-1} = \cup(x_n, y_n)$ with $y_n = x_n + 1$, then $m(E^{-1}) = \infty$ but $E = (y_n^{-1}, x_n^{-1})$ can have 0 as a point of dispersion. (If $x_k = 3^k$, for example, $\lim_K \Sigma_{k=K}^{\infty}(3^{-k} - (3^k+1)^{-1}/3^{-K} \leq \lim_K \Sigma_{k=K}^{\infty} 9^{-k} \cdot 3^K = \lim_K 9^{-K}(9/8)3^K = 0$ shows that the resulting E has $D_E^r(0) = 0$.)

The definition of Pr' could have been obtained in another way be considering a 'non-homogeneous' Poisson process whose probabilities, $(g)Pr$, are defined by

$$(g)Pr(\{S : \#(S \cap A) = n\}) = e^{-c(A)}c(A)^n/n!$$

where for measurable sets A, $c(A) = \int_A g(t)dt$. (See [2]). Here g is assumed to be a non-negative measurable function which is integrable on each closed interval contained in $(0, \infty)$. Also, for pairwise disjoint measurable sets A_k and non-negative integers n_k,

$$(g)Pr(\{S : \#(S \cap A_k) = n_k,\ k = 1, 2, \ldots\}) = \Pi_k e^{-c(A_k)}c(A_k)^{n_k}/n_k!$$

When $g(t) = c$ one obtains the usual Poisson process. When $g(t) = t^{-2}$, one has for sets $E \subset (0, 1)$,

$$Pr'(\{S : \#(S \cap E) = n\}) = (t^{-2})Pr(\{S : \#(S \cap E) = n\}).$$

Note also that

$$1 = \lim_{h\to 0}(t^{-2})Pr(\{S : S \cap E \cap (0,h) = \phi\} = \lim_{h\to 0} e^{-c(E\cap(0,h))}$$

iff $0 = \lim_{h\to 0} \int_{E\cap(0,h)} t^{-2}dt$

iff there is an open set $G \supset E$ with $G = \cup(a_i, b_i)$

so that $\lim_{h\to 0} \Sigma_h \frac{b_i - a_i}{a_i b_i} = 0$.

We now examine other probabilities determined by finite functions g which are non-increasing on $(0,1)$ and satisfy $\int_0^1 g(t)dt = \infty$. These conditions guarantee that with probability 1 sequences will have 0 as a limit point and have no other limit point in $(0, 1)$.

We first consider $g(t) = t^{-1}$. Given a measurable set E, $(t^{-1})Pr(\{S : \#(S \cap E) = 0\}) = e^{-\int_E t^{-1}dt}$. Thus

$\lim_{h\to 0}\ Pr(\#(S \cap E \cap (0,h)) = 0) = 1$

iff $\lim_{h\to 0}\ e^{-\int_{E\cap(0,h)} t^{-1}dt} = 1$

iff $\lim_{h\to 0}\ \int_{E\cap(0,h)} t^{-1}dt = 0$

iff there is $G = \cup(a_i, b_i)$ with EG so that

$\lim_{h\to 0} \Sigma_h \ell n(b_n/a_n) = 0$

iff $\lim_N\ \Pi_N^{\infty} \frac{b_n}{a_n} = 1$.

But $\Pi \frac{b_n}{a_n}$ converges iff $\Pi \frac{a_n}{b_n}$ converges iff $\Sigma \frac{a_n}{b_n} - 1$ converges. (See [1], p. 96.) That is, this holds iff $\lim_{h \to 0} \Sigma_h \frac{b_n - a_n}{b_n} = 0$. Thus, if E is contained in an open set $G = \cup(a_n, b_n)$, so that $\lim_{h\to 0} \Sigma_h \frac{b_n - a_n}{b_n} = 0$, then $D^r_E(0) = 0$ since for N defined by $h \in (a_N, b_N)$, $\lim_{h\to 0} \Sigma_h \frac{b_n - a_n}{b_N} \le \lim_{h \to 0} \Sigma_h \frac{b_n - a_n}{b_n}$.

The following example shows that there are sets E with $D_E(0) = 0$ but $\int_E 1/t\, dt = \infty$.

Example. Let $E = \cup_n(3^{-n}(1-n^{-1}), 3^{-n})$. These intervals are distinct because $3^{-n}(1 - 1/n) - 3^{-(n+1)} = (3(1 - 1/n) - 1)/3^{n+1} = (2 - 3/n)/3^{n+1} > 0$. Then $D^r_E(0) \le \lim \Sigma^{\infty}_{n=N} 3^{-n}(1/n)/3^{-N} < \lim_N 2 \cdot 3^{-N}(1/N)/3^{-N} = \lim_N 2/N = 0$. However, $\Sigma(b_n - a_n)/b_n = \Sigma_n n^{-1} 3^{-n}/3^{-n} = \Sigma n^{-1} = \infty$.

It is tempting at this point to consider functions g the values of which are smaller than $1/t$ for every $t \in (0, 1)$ and $\int_0^1 g(t)dt = \infty$ in order to obtain a better process for estimating density 0 and approximate continuity at 0 from the right. However, no such better process of this type exists. To see this we show that functions $g(t)$, whose ratio to $1/t$ approaches 0 on a sequence t_n decreasing to 0, have sets E of upper density 1 at 0 so that $(g)Pr(\{S : S \cap E \text{ is infinite}\}) = 0$.

We now characterize this situation.

Theorem 3. *Suppose g is a non-increasing function on $(0,1)$ and $\int_0^1 g(t)dt = \infty$. Then $\underline{\lim}_{t\to 0} t \cdot g(t) = 0$ iff there is a set E with $\bar{D}^r_E(0) = 1$ (alternately, $\bar{D}^r_E(0) > 0$) so that $\int_E g(t)dt < \infty$ and thus $(g)Pr(\{S : \#(S \cap E) = \infty\}) = 0$.*

Proof. Suppose $g(t)$ satisfies $\underline{\lim}\, t \cdot g(t) = 0$. Choose a sequence $\{t_n\}$ decreasing to 0 so that $t_n g(t_n) \le 3^{-n}$. Let $E = \cup\, (t_n, 2^n t_n)$. Then $\bar{D}^r_E(0) \ge \lim_n\, (2^n t_n - t_n)/2^n t_n = 1$ but

$$\int_E g(t)dt \le \Sigma g(t_n)(2^n t_n - t_n) \le \Sigma g(t_n) \cdot t_n 2^n \le \Sigma 2^n/3^n < \infty.$$

It follows that $(g)Pr(\{S : \#(S \cap E) = \infty\}) = 0$ because

$$\lim_{h\to 0}(g)Pr(\{S : \#(S \cap E \cap (0,h)) = 0\} = \lim_{h \to 0+} e^{-\int_{E\cap(0,h)} g(t)dt} = 1.$$

To see that the converse holds, note that if $\underline{\lim}\, g(t) \cdot t > 0$ there is $\varepsilon > 0$ so that $g(t) \cdot t > \varepsilon$ when $t < \varepsilon$ and thus for any set E, $\int_{E\cap(0,\varepsilon)} g(t)dt \ge \varepsilon \cdot \int_{E\cap(0,\varepsilon)} 1/t\, dt$. Then, if $\int_E g(t)dt < \infty$, $\int_E 1/t\, dt < \infty$ and E has 0 as a point of dispersion according to the calculations for the function $1/t$.

A simple example of such a function g with $\underline{\lim}\, tg(t) = 0$ and $\int_0^\infty g(t)dt = \infty$ is $g(t) = -1/(t \cdot \ell n t)$ if $t \in [0, e^{-1})$; $g(t) = e$ if $t \in [e^{-1}, 1)$.

We now relate sets E and functions $g > 0$ which are non-increasing on $(0,1)$ and have $\int_0^1 g(t)dt = \infty$ and $\int_E g(t)dt < \infty$. A consequence of this theorem is that $\underline{D}^r_E(0) = 0$ iff there is a $g > 0$ non-increasing on $(0,1)$ with $\int_E g(t)dt < \infty$ and $\int_0^1 g(t)dt = \infty$ so that $(g)Pr(\{S : \#(S \cap E) = \infty\}) = 0$.

Theorem 4. *For any measurable set E there is a non-increasing function $g > 0$ on $(0,1)$ with $\int_0^1 g(t)dt = \infty$ and $\int_E (g(t)dt < \infty$ iff $\underline{D}^r_E(0) = 0$.*

Proof. Suppose E has $\underline{D}^r_E(0) = 0$. Let $t_0 = 1$ and for $n > 0$ let t_n satisfy $t_n < t_{n-1}/2$ and $m(E \cap (0, t_n)) \leq 2^{-n} t_n$. Let $g(t) = (t_{n-1} - t_n)^{-1}$ if $t \in [t_n, t_{n-1})$. Then g is non-increasing on $(0,1)$ and $\int_0^1 g(t)dt = \Sigma(t_{n-1} - t_n)^{-1} \cdot (t_{n-1} - t_n) = \infty$. Also,

$$\begin{aligned}
\int_E g(t)dt &= \sum_{n=0}^{\infty} (t_{n-1} - t_n)^{-1} m(E \cap (t_n, t_{n-1})) \\
&\leq \sum_{n=0}^{\infty} (t_{n-1} - t_n)^{-1} \cdot 2^{-n} t_{n-1} + g(t_1)(1 - t_1) \\
&\leq \sum_{n=0}^{\infty} 2^{-n} + g(t_1)(1 - t_1) < \infty.
\end{aligned}$$

To see the converse, we show that if $\underline{D}^r_E(0) > 0$ and $g > 0$ is non-increasing on $(0,1)$ with $\int_0^1 g(t)dt = \infty$. Then $\int_E g(t)dt$ also equals ∞. Choose any $\{x_n\}$ decreasing to 0 so that for $0 < x < x_1$, $m(E \cap (0,x)) > \underline{D}^r_E(0)/2 = \varepsilon$. Determine a sequence $\{t_n\}$ decreasing to 0 so that $x_1 = t_1$, $x_k = t_{n_k}$ and

$$\int_{x_k}^{x_{k+1}} g(t)dt - \sum_{i=n_k}^{n_{k+1}} g(t_i)(t_{i-1} - t_i) < 2^{-k}.$$

Then

$$(E \cap (0, t_1)) \times (0, g(t_1)) \cup \bigcup_{n=1}^{\infty} (E \cap (0, t_n)) \times (g(t_{n-1}), g(t_n))$$

is a pairwise disjoint union of sets whose points lie in the plane under the graph of g on E. Then $\int_E g(t)dt$ is greater than or equal to the sum of the measures of these sets. That is, $\varepsilon^{-1} \int g(t)dt \geq g(t_1)t_1 + (g(t_2) - g(t_1))t_2 + \cdots + (g(t_{n+1}) - g(t_n))t_n + \cdots \geq g(t_1)(t_1 - t_2) + g(t_2)(t_2 - t_3) + \cdots + g(t_n)(t_n - t_{n+1}) + \cdots = \infty$. We note that both the earlier probability spaces defined by $\{t_n\}$ decreasing to 0 with $t_{n+1} > rt_n$ for some $r > 0$ and the spaces with $\underline{\lim}\, t \cdot g(t) > 0$ give rise to stronger notion of 'point of dispersion' and hence to a weaker topology than the density topology if a set E is defined to be open if each point of E is a 'point of dispersion' of E^c. We also note the following problem:

Problem. Characterize the measurable sets E (with $D_E^r(0) = 0$) for which there is a sequence $\{t_n\}$ decreasing to 0 and $r > 0$ so that $t_{n+1} > rt_n$ and for almost every x in $(0,1)$, $C_E(xt_n)$ approaches 0.

References

[1] K. Knopp, Infinite Sequences and Series, Dover Publications, 1956.

[2] S. Ross, Stochastic Processes, John Wiley and Sons, 1983.

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