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## $A_\infty$ TYPE CONDITIONS FOR GENERAL MEASURES IN $\mathbf{R}^1$

**ABSTRACT.** Given two Borel measures  $\mu, \nu$  in  $\mathbf{R}^1$ , finite on compacts, we deal with the conditions  $\nu \in A_\infty(\mu)$  and  $\nu \in A_\infty^+(\mu)$ . We point out several characterizing statements, expressed in terms of maximal functions,  $A_p$  classes, reverse Hölder inequalities, exponential type conditions, and their symmetric versions. We use ideas from [1] and [19] and extend their results to this rather general context (in the one-sided case we assume that  $\mu$  and  $\nu$  are mutually absolutely continuous). We consider an application to the Gehring lemma.

### 1. INTRODUCTION

In 1972, B. Muckenhoupt introduced the  $A_p$  classes of weights. He has shown [20] that a weight  $w$  in  $\mathbf{R}^n$  belongs to  $A_p$ ,  $1 < p < \infty$ , that is,

$$\frac{1}{|Q|} \int_Q w \left( \frac{1}{|Q|} \int_Q w^{1-p'} \right)^{p-1} \leq K ,$$

if and only if the *Hardy-Littlewood maximal operator*

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| dy ,$$

while  $w \in A_1$ , that is,  $Mw \leq Kw$ , if and only if  $M$  is of the weak type (1,1) with respect to  $w$ ,

$$\lambda \int_{\{Mf > \lambda\}} w \leq K \int |f| w .$$

The class  $A_\infty$ , formed by the weights  $w$  satisfying

$$(1.1) \quad \frac{w(E)}{w(Q)} \leq K \left( \frac{|E|}{|Q|} \right)^\delta, \quad E \subset Q, \quad E \text{ measurable},$$

where  $K$  and  $\delta$  do not depend on  $E$  and  $Q$ , and  $w(Q) = \int_Q w \, dx$ , was extensively studied in [7] and [21]. Both the endpoints of the  $A_p$  scale, the classes  $A_1$  and  $A_\infty$ , are of exceptional significance. While there is a gap between  $A_1$  and  $\bigcap_{p>1} A_p$  (see e.g. [5], [11], [15], [16], [22]), one gets a different picture at the opposite endpoint:  $A_\infty$  is exactly  $\bigcup_{p>1} A_p$  ([21], [7]). A lot of equivalent definitions of  $A_\infty$  are known (see e.g. [7], [14], [10], [11], [8]), and important applications of  $A_\infty$  have been pointed out ([4], [2], [6]).

While nearly the entire  $A_p$  theory has been translated to its one-sided (and one-dimensional) analog,  $A_p^+$  theory, initiated by the investigation of the *one-sided maximal operator*

$$M^+ f(x) = \sup_{h>0} \frac{1}{h} \int_x^{x+h} |f|$$

([24], [17], [18]), the corresponding one-sided analog of  $A_\infty$ , the class  $A_\infty^+$ , was introduced only recently in [19].

One of the most important properties of  $A_\infty$  is its equivalence to the *reverse Hölder (RH) inequality*,

$$\left( \frac{1}{|Q|} \int_Q w^{1+\delta} \right)^{1/(1+\delta)} \leq K \frac{1}{|Q|} \int_Q w ,$$

$K, \delta$  independent of  $Q$  (see [7], [11]). The question what is the corresponding one-sided analog of the RH inequality was solved by F.J.Martín-Reyes [17], and called the *weak RH inequality*; he proved that if  $w \in A_p^+$ ,  $1 < p < \infty$ , then there exist  $K, \delta$  such that

$$\int_a^b w^{1+\delta} \leq K \int_a^b w [M^-(w\chi_{(a,b)})(b)]^\delta ,$$

and used this result to show that  $w \in A_p^+$  implies  $w \in A_{p-\epsilon}^+$ . In [19] it was shown that the last inequality is equivalent to  $w \in A_\infty^+$  (in a somewhat more general context).

Let us turn attention to measures. Coifman and Fefferman [7] studied  $A_\infty$  and RH type conditions for doubling measures. Recall that  $\mu$  is said to be *doubling* if  $\mu(2Q) \leq K\mu(Q)$ , where  $2Q$  is a cube concentric with  $Q$  but with sides twice as long. If we define  $\nu \in A_\infty(\mu)$  by

$$\frac{\mu(E)}{\mu(Q)} \leq K \left( \frac{\nu(E)}{\nu(Q)} \right)^\delta, \quad E \subset Q, \quad E \text{ measurable},$$

$K, \delta$  independent of  $E, Q$ , it follows from their result that the symmetry relation

$$(1.2) \quad \nu \in A_\infty(\mu) \quad \Leftrightarrow \quad \mu \in A_\infty(\nu)$$

holds and that  $\nu \in A_\infty(\mu)$  is equivalent to the RH inequality

$$\left( \frac{1}{\mu(Q)} \int_Q \left( \frac{d\nu}{d\mu} \right)^{1+\delta} d\mu \right)^{1/(1+\delta)} \leq \frac{K}{\mu(Q)} \int_Q \frac{d\nu}{d\mu} d\mu,$$

$K, \delta$  independent of  $Q$ . The proof essentially relies on the Calderón–Zygmund decomposition lemma and entails therefore to assume the doubling condition. A comprehensive expository is given in [26].

In Theorem 4.4 below we show that the symmetry relation (1.2) holds for general measures, too. Thus, our definition of  $\nu \in A_\infty(\mu)$  is consistent with (1.1).

Translated to the one-sided case (1.2) naturally turns to the “anti-symmetry” relation

$$\nu \in A_\infty^+(\mu) \quad \Leftrightarrow \quad \mu \in A_\infty^-(\nu),$$

as shown in [19] for the case when  $\mu$  and  $\nu$  are weighted measures, that is,  $d\mu(x) = g(x)dx$  and  $d\nu(x) = w(x)dx$  with positive  $w$  and  $g$ .

If we omit the doubling condition, we meet surprisingly difficult obstacles (cf. [25], [26], [9], [3], [1]). For example, Sjögren [25] showed that while the maximal operator

$$M_\mu f(x) = \sup_{Q \ni x} \frac{1}{\mu(Q)} \int_Q |f(y)| d\mu(y)$$

is in  $\mathbf{R}^1$  always of weak type (1,1) with respect to  $\mu$ , this is no longer true in  $\mathbf{R}^n$ ,  $n > 1$ , even not for  $\mu$  absolutely continuous with respect to the Lebesgue measure.

The fundamental paper of Andersen [1] develops a method allowing to extend the  $A_p$  and  $A_p^+$  theory to the context of general measures in  $\mathbf{R}^1$ .

As far as we know, no attempts have been made to obtain results on  $A_\infty^+$  or  $A_\infty$  type conditions for general measures. This is a subject of the present note. Our aim is to look backwards at the methods used by different authors and to point out how the theory rounds off, and not so much to obtain new results. For illustration, let us indicate the sources of our main results. Theorems 3.3 and 3.4 below were shown in [19] in the case of weighted measures (thus, in particular, absolutely continuous with respect to Lebesgue measure). The statements of Theorem 4.4 have been subsequently proved to be equivalent in [7], [10], [14] and [13] under the doubling condition, see also [26]. We shall show (Section 3) how the one-sided theory works with general measures and afterwards (Section 4) use the results obtained to prove the two-sided theorems. It follows from the proof of Theorem 4.4 that the symmetry relation (1.2) can be proved (in  $\mathbf{R}^1$ ) assuming merely that  $\mu$  and  $\nu$  are Borel measures finite on compact sets, i.e., without the doubling condition or any absolute continuity. This yields some interesting consequences; for example, it sheds light on the relation between two conditions of exponential-logarithmic type, which are both equivalent to  $A_\infty$ : the Hruščev condition

$$\sup_Q \left( \frac{1}{|Q|} \int_Q w \right) \exp \left( \frac{1}{|Q|} \int_Q \log \left( \frac{1}{w} \right) \right) \leq K ;$$

and the Fujii condition

$$\sup_Q \frac{1}{w(Q)} \int_Q \log^+ \left( \frac{w(x)}{w_Q} \right) w(x) dx \leq K ,$$

where  $w_Q = w(Q)/|Q|$ . The equivalence between these two conditions, which is not quite obvious, was pointed out in [13], but again under the doubling condition.

Further, we show that  $A_\infty$  is equivalent to the reverse Hölder inequality, again using the one-sided results instead of the Calderón-Zygmund theory. Here, the main disadvantage of our approach appears again: it does not apply to the higher-dimensional case.

Finally, we prove (Theorem 4.5) the analog of Gehring's lemma [12].

As usual,  $K$  will always denote absolute constant, independent of appropriate quantities. We write  $(a, b)$  for an open interval,  $[a, b]$  for a closed interval and  $\{a, b\}$  for an interval which might be either open or closed. We shall write  $E$  for a Borel set in  $\mathbf{R}$ . The products of type  $0 \cdot \infty$  will be taken to be zero.

## 2. DEFINITIONS AND PRELIMINARY RESULTS.

Let  $\mu$  be a nonnegative Borel measure in  $\mathbf{R}$ , finite on compact sets (as known, then  $\mu$  is regular [23], Theorem 2.18). We put

$$\begin{aligned} M_{\mu}^{+} f(x) &= \sup_{h>0} \frac{1}{\mu[x, x+h]} \int_{[x, x+h]} |f| d\mu ; \\ M_{\mu}^{-} f(x) &= \sup_{h>0} \frac{1}{\mu(x-h, x]} \int_{(x-h, x]} |f| d\mu ; \\ M_{\mu} f(x) &= \sup_{I \ni x} \frac{1}{\mu(I)} \int_I |f| d\mu , \end{aligned}$$

where  $I$  denotes an interval in  $\mathbf{R}$  (cf. [1]).

**2.1. Definition.** Let  $w$  be a weight (nonnegative  $\mu$ -measurable function) in  $\mathbf{R}^1$ . We say that  $w \in A_p^+(\mu)$ ,  $p \in (1, \infty)$ , if

$$(2.1) \quad \left( \int_{(a,b]} w d\mu \right)^{1/p} \left( \int_{(b,c)} w^{1-p'} d\mu \right)^{1/p'} \leq K\mu(a, c)$$

with some  $K$  independent of  $a < b < c$ , where  $p' = p/(p-1)$ .

We say that  $w \in A_1^+(\mu)$  if  $M_{\mu}^{-} w(x) \leq Kw(x)$  for  $\mu$ -almost all  $x$ . Similarly,  $w \in A_p^-(\mu)$ ,  $p \in (1, \infty)$ , if

$$(2.2) \quad \left( \int_{(b,c)} w d\mu \right)^{1/p} \left( \int_{(a,b]} w^{1-p'} d\mu \right)^{1/p'} \leq K\mu(a, c)$$

and  $w \in A_1^-(\mu)$  if  $M_{\mu}^{+} w(x) \leq Kw(x)$  for  $\mu$ -almost all  $x$ .

We say that  $w \in A_p(\mu)$ ,  $p \in (1, \infty)$ , if

$$\left( \int_I w d\mu \right)^{1/p} \left( \int_I w^{1-p'} d\mu \right)^{1/p'} \leq K\mu(I)$$

for some  $K$  independent of interval  $I$  in  $\mathbf{R}$ .

We say that  $w \in A_1(\mu)$  if  $M_{\mu} w \leq Kw$   $\mu$ -almost everywhere.

The following theorem is due to Andersen [1].

**2.2. Theorem.** *Let  $p > 1$ . Then the following statements are equivalent.*

- (i) 
$$\int (M_{\mu}^{+} f)^p w d\mu \leq K \int |f|^p w d\mu ;$$
- (ii) 
$$w \in A_p^{+}(\mu) ;$$
- (iii) 
$$w^{1-p'} \in A_{p'}^{-}(\mu) ;$$
- (iv) 
$$\int (M_{\mu}^{-} f)^{p'} w^{1-p'} d\mu \leq K \int |f|^{p'} w^{1-p'} d\mu .$$

**2.3. Remark.** *It is easy to see that  $A_p(\mu) = A_p^{+}(\mu) \cap A_p^{-}(\mu)$ ,  $p \geq 1$ , and that*

$$\max \{M_{\mu}^{-} f, M_{\mu}^{+} f\} \leq M_{\mu} f \leq M_{\mu}^{+} f + M_{\mu}^{-} f .$$

*Therefore,  $w \in A_p(\mu)$ ,  $p > 1$ , if and only if*

$$\int (M_{\mu} f)^p w d\mu \leq K \int |f|^p w d\mu .$$

By the same argument as in [18] with trivial changes we can prove the following factorization theorem (we have only to realize that  $[M_{\mu}^{+} (|f|^{\alpha})]^{1/\alpha}$  and  $[M_{\mu}^{-} (|f|^{\alpha})]^{1/\alpha}$  are sublinear operators for  $\alpha > 1$ , and use (Andersen's) Theorem 2.2.)

**2.4. Theorem.** *Let  $p > 1$ . Then*

$$A_p^{+}(\mu) = A_1^{+}(\mu) \cdot [A_1^{-}(\mu)]^{1-p} ,$$

*that is, each  $w \in A_p^{+}(\mu)$  can be factorized as  $w = w_0 \cdot w_1^{1-p}$ , where  $w_0 \in A_1^{+}(\mu)$ ,  $w_1 \in A_1^{-}(\mu)$ . Similarly,*

$$A_p^{-}(\mu) = A_1^{-}(\mu) \cdot [A_1^{+}(\mu)]^{1-p} ,$$

*and*

$$A_p(\mu) = A_1(\mu) \cdot [A_1(\mu)]^{1-p} .$$

### 3. ONE-SIDED PROBLEMS.

Let  $\mu, \nu$  be nonnegative Borel measures in  $\mathbf{R}$ , finite on compact sets, and let moreover  $\mu$  be absolutely continuous with respect to  $\nu$  and vice versa.

**3.1. Definition.** We say that  $\nu \in A_{\infty}^+(\mu)$  if there exist  $K, \delta$  positive such that for all  $a < b < c$  and  $E \subset [b, c)$

$$\frac{\mu(E)}{\mu(a, c)} \leq K \cdot \left( \frac{\nu(E)}{\nu(a, b]} \right)^{\delta}.$$

**3.2. Remark.** Obviously,  $\nu \in A_{\infty}^+(\mu)$  implies  $\mu \ll \nu$ . However, unlike in the two-sided case, as we shall see,  $\frac{d\mu}{d\nu}$  can vanish on a set of positive measure (for example, put  $d\nu(x) = dx$  and  $d\mu(x) = \chi_{[0, \infty)}(x) dx$ ). Thus,  $\nu \in A_{\infty}^+(\mu)$  does not imply  $\nu \ll \mu$ . But, since we want to study natural relationship between  $\nu \in A_{\infty}^+(\mu)$  and  $\mu \in A_{\infty}^-(\nu)$  (which, of course, would imply  $\nu \ll \mu$ ), it is only reasonable to avoid trivial cases by assuming that  $\mu$  and  $\nu$  are mutually absolutely continuous (note that they need not be absolutely continuous e.g. with respect to Lebesgue measure).

In what follows we put  $w(x) = \frac{d\nu}{d\mu}(x)$ . We know from absolute continuity that such  $w$  exists and  $0 < w(x) < \infty$   $\mu$ -almost everywhere. We shall often write  $w \in A_{\infty}^+(\mu)$  instead of  $\nu \in A_{\infty}^+(\mu)$ .

The next two theorems were proved in [19] in the case of weighted measures.

**3.3. Theorem.** The following statements are equivalent.

- (i) There exists  $p > 1$  such that  $w \in A_p^+(\mu)$ ;
- (ii)  $w \in A_{\infty}^+(\mu)$ ;
- (iii) for any  $\alpha \in (0, 1)$  there exists  $\beta \in (0, 1)$  such that for all  $a \leq b < c$  and  $E \subset [b, c)$

$$\frac{\nu(E)}{\nu[a, b]} < \beta \quad \text{implies} \quad \frac{\mu(E)}{\mu[a, c]} < \alpha;$$

- (iv) for every  $\alpha \in (0, 1)$  there exists  $\beta \in (0, 1)$  such that the following implication holds: whenever  $\lambda > 0$  and  $a < b$  are such that

$$(3.1) \quad \lambda \leq \frac{\nu\{a, x\}}{\mu\{a, x\}} \quad \text{for all } x \in (a, b);$$

then

$$\mu(\{x \in \{a, b\}; w(x) > \beta\lambda\}) > \alpha\mu\{a, b\};$$

(v) for every  $\alpha \in (0, 1)$  there exists  $\beta \in (0, 1)$  such that the following implication holds: whenever  $\lambda > 0$  and  $a < b$  are such that (3.1) holds and

$$\frac{\nu\{a, b\}}{\mu\{a, b\}} \leq 2\lambda,$$

then

$$\mu(\{x \in \{a, b\}; w(x) > \beta\lambda\}) > \alpha\mu\{a, b\};$$

(vi) there exist  $K, \delta > 0$  such that for all  $a < b$

$$\int_{(a, b]} w^{1+\delta} d\mu \leq K \int_{(a, b]} w d\mu \cdot [M_\mu(w\chi_{(a, b)})(b)]^\delta;$$

(vii) there exist  $K, \delta > 0$  such that for all  $a < b$

$$M_\nu(w^\delta \chi_{(a, b)})(b) \leq K [M_\mu(w\chi_{(a, b)})(b)]^\delta;$$

(viii) there exists  $p > 1$  such that  $\frac{1}{w} \in A_p^-(\nu)$ ;

(ix) there exists  $K$  such that for all  $a < b < c$  satisfying  $\mu(a, b) \leq \frac{1}{2}\mu(a, c) \leq \mu(a, b]$

$$\frac{\nu(a, b]}{\mu(a, b]} \cdot \exp\left(\frac{1}{\mu[b, c]} \int_{[b, c]} \log \frac{1}{w} d\mu\right) \leq K;$$

(x)  $\frac{1}{w} \in A_\infty^-(\nu)$ ;

(xi) for any  $\alpha \in (0, 1)$  there exists  $\beta \in (0, 1)$  such that for all  $a < b \leq c$  and  $E \subset (a, b]$

$$\frac{\mu(E)}{\mu[b, c]} < \beta \quad \text{implies} \quad \frac{\nu(E)}{\nu(a, c]} < \alpha;$$

(xii) for every  $\alpha \in (0, 1)$  there exists  $\beta \in (0, 1)$  such that the following implication holds: whenever  $\lambda > 0$  and  $a < b$  are such that

$$(3.2) \quad \lambda \leq \frac{\mu(x, b]}{\nu(x, b]} \quad \text{for all } x \in [a, b),$$

then

$$\nu\left(\left\{x \in (a, b); \frac{1}{w(x)} > \beta\lambda\right\}\right) > \alpha\nu(a, b];$$

(xiii) for every  $\alpha \in (0, 1)$  there exists  $\beta \in (0, 1)$  such that the following implication holds: whenever  $\lambda > 0$  and  $a < b$  are such that (3.2) holds and

$$\frac{\mu(a, b]}{\nu(a, b]} \leq 2\lambda,$$

then

$$\nu \left( \left\{ x \in (a, b); \frac{1}{w(x)} > \beta\lambda \right\} \right) > \alpha\nu(a, b);$$

(xiv) there exist  $K, \delta > 0$  such that for all  $a < b$

$$\int_{[a, b]} w^{-(1+\delta)} d\nu \leq K \int_{[a, b]} \frac{1}{w} d\nu \cdot \left[ M_\nu \left( \frac{1}{w} \chi_{[a, b]} \right) (a) \right]^\delta;$$

(xv) there exist  $K, \delta > 0$  such that for all  $a < b$

$$M_\mu \left( \left( \frac{1}{w} \right)^\delta \chi_{[a, b]} \right) (a) \leq K \left[ M_\nu \left( \frac{1}{w} \chi_{[a, b]} \right) (a) \right]^\delta;$$

(xvi) there exists  $K$  such that for all  $a < b < c$  satisfying  $\nu(b, c) \leq \frac{1}{2}\nu(a, c) \leq \nu[b, c]$

$$\frac{\mu[b, c]}{\nu[b, c]} \cdot \exp \left( \frac{1}{\nu(a, b]} \int_{(a, b]} \log w d\nu \right) \leq K.$$

*P r o o f.* (i) $\Rightarrow$ (ii) can be proved completely in the same way as in [19], that is, using Hölder's inequality and  $A_p^+$ ,

$$(\mu(E))^p \leq \nu(E) \left( \int_E w^{1-p'} d\mu \right)^{p-1} \leq K(\mu(a, c))^p \frac{\nu(E)}{\nu(a, b]},$$

which is  $w \in A_\infty^+(\mu)$  with  $\delta = 1/p$ .

(ii) $\Rightarrow$ (iii) is easy; we have only to realize that  $w \in A_\infty^+(\mu)$  is equivalent to

$$\frac{\mu(E)}{\mu[a, c]} \leq K \cdot \left( \frac{\nu(E)}{\nu[a, b]} \right)^\delta, \quad a \leq b < c, \quad E \subset [b, c].$$

This follows by the usual limiting argument.

(iii) $\Rightarrow$ (iv): Let  $\lambda$  and  $\{a, b\}$  be as in (iv). We put  $x_0 = b$  and, for any negative integer  $k$ ,

$$x_k = \inf \{ x \in \{a, b\}; 2^k \nu\{a, b\} \leq \nu\{a, x\} \}.$$

Now, (3.1) yields that  $\lim_{k \rightarrow -\infty} x_k = a$ , and the sequence  $\{x_k\}$  is (not necessarily strictly) decreasing. It follows from the definition of  $\{x_k\}$  that

$$(3.3) \quad \nu\{a, x_k\} \leq 2^k \nu\{a, b\} \leq \nu\{a, x_k\}.$$

Put  $\Gamma = \{k \in -\mathbf{N}; x_k \neq x_{k+1}\}$ , then  $\{a, b\} = \bigcup_{k \in \Gamma} [x_k, x_{k+1}]$  (we note that if  $\{a, b\} = [a, b]$ ,  $\Gamma$  is finite).

We claim that for any  $k$

$$\nu\{a, x_{k+1}\} \leq 4\nu[x_{k-1}, x_k].$$

Indeed, using twice (3.3), we get

$$\begin{aligned} \nu\{a, x_{k+1}\} &\leq 2^{k+1}\nu\{a, b\} = 4(2^k - 2^{k-1})\nu\{a, b\} \\ &\leq 4(\nu\{a, x_k\} - \nu\{a, x_{k-1}\}) = 4\nu[x_{k-1}, x_k]. \end{aligned}$$

Thus

$$\lambda \leq 4 \frac{\nu[x_{k-1}, x_k]}{\mu\{a, x_{k+1}\}},$$

and we can estimate the measure of

$$E_\beta = \{x \in \{a, b\}; w(x) \leq \beta\lambda\}$$

in the following way: For any  $k$ ,

$$\begin{aligned} \frac{\nu(E_\beta \cap [x_k, x_{k+1}])}{\nu[x_{k-1}, x_k]} &\leq \frac{\beta\lambda\mu(E_\beta \cap [x_k, x_{k+1}])}{\nu[x_{k-1}, x_k]} \\ &\leq 4\beta \frac{\mu(E_\beta \cap [x_k, x_{k+1}])}{\mu\{a, x_{k+1}\}} < 4\beta. \end{aligned}$$

Given  $\gamma \in (0, 1)$ , we can (by (iii)) choose  $\beta$  so that

$$\frac{\mu(E_\beta \cap [x_k, x_{k+1}])}{\mu[x_{k-1}, x_{k+1}]} \leq \gamma.$$

Therefore

$$\begin{aligned} \mu(\{x \in \{a, b\}; w(x) > \beta\lambda\}) &= \sum_{k \in \Gamma} \mu([x_k, x_{k+1}] \setminus E_\beta) \\ &\geq \sum_{k \in \Gamma} \mu[x_k, x_{k+1}] - \gamma \sum_{k \in \Gamma} \mu[x_{k-1}, x_{k+1}] \\ &\geq \mu\{a, b\} \cdot (1 - 3\gamma), \end{aligned}$$

since each point of  $\{a, b\}$  belongs at most to 3 intervals of type  $[x_{k-1}, x_{k+1}]$ ,  $k \in \Gamma$ .

(iv) $\Rightarrow$ (v) is obvious.

(v) $\Rightarrow$ (vi):

Define  $\lambda_0 = M_\mu^-(w\chi_{(a,b)})(b)$  and take  $\lambda > \lambda_0$ . Then (cf. [28], Theorem 1.3.8, and [1], proof of Theorem 1)

$$\{x \in (a, b]; w(x) > \lambda\} \subset \{M_\mu^-(w\chi_{(a,b)}) > \lambda\} = \bigcup_j \{a_j, b_j\},$$

where  $\{a_j, b_j\} \subset (a, b]$  for all  $j$  and

$$\lambda \leq \frac{\nu\{a_j, x\}}{\mu\{a_j, x\}}$$

for all  $x \in (a_j, b_j]$ .

We fix  $j$  and define  $x_0 = a_j$ ,

$$A_1 = \{x \in \{a_j, b_j\}; \frac{\nu\{a_j, x\}}{\mu\{a_j, x\}} \leq 2\lambda\},$$

and  $x_1 = \sup A_1$  if  $A_1 \neq \emptyset$ , and  $x_1 = x_0$  if  $A_1 = \emptyset$ . Suppose that  $x_{n-1}$  is chosen. If  $x_{n-1} = b_j$ , we stop the process. If  $x_{n-1} < b_j$ , we put

$$A_n = \{x \in [x_{n-1}, b_j); \frac{\nu[x_{n-1}, x]}{\mu[x_{n-1}, x]} \leq 2^n \lambda\}$$

and  $x_n = \sup A_n$  if  $A_n \neq \emptyset$  and  $x_n = x_{n-1}$  if  $A_n = \emptyset$ .

Now, let  $n(j)$  be the least integer  $k$  such that  $x_k > a_j$ , and put

$$\Gamma(j) = \{k \in \mathbf{N} \setminus \{n(j)\}; x_k < x_{k+1}\}.$$

Observe that  $x_n \nearrow b_j$ , and so

$$\{a_j, b_j\} = \{a_j, x_{n(j)}\} \cup \bigcup_{n \in \Gamma(j)} [x_n, x_{n+1}).$$

Moreover, the following estimates are true:

$$\frac{\nu\{a_j, x_{n(j)}\}}{\mu\{a_j, x_{n(j)}\}} \leq 2^{n(j)} \lambda;$$

$$2^{n(j)-1} \lambda \leq \frac{\nu\{a_j, x\}}{\mu\{a_j, x\}} \quad x \in (a_j, x_{n(j)}];$$

and, for  $n \geq n(j)$  and  $n \in \Gamma(j)$ ,

$$\frac{\nu[x_n, x_{n+1}]}{\mu[x_n, x_{n+1}]} \leq 2^{n+1} \lambda ;$$

and

$$2^n \lambda < \frac{\nu[x_n, x]}{\mu[x_n, x]} \quad x \in (x_n, x_{n+1}] .$$

Only the last estimate needs proof, the preceding three follow immediately from the definition of  $x_n$  and the  $\sigma$ -additivity of  $\mu$ . If  $x \in (x_n, x_{n+1}]$ , we have

$$2^n \lambda \mu[x_{n-1}, x] < \nu[x_{n-1}, x]$$

and therefore

$$\nu[x_n, x] = \nu[x_{n-1}, x] - \nu[x_{n-1}, x_n] > 2^n \lambda \mu[x_n, x] .$$

Now, using the estimates just proved and (v) we obtain

$$\begin{aligned} \nu\{a_j, x_{n(j)}\} &\leq 2^{n(j)} \lambda \mu\{a_j, x_{n(j)}\} \\ &< \alpha^{-1} 2^{n(j)} \lambda \mu\{x \in \{a_j, x_{n(j)}\}; w(x) > 2^{n(j)-1} \beta \lambda\} \end{aligned}$$

and, for  $n \in \Gamma(j)$ ,

$$\begin{aligned} \nu[x_n, x_{n+1}] &\leq 2^{n+1} \lambda \mu[x_n, x_{n+1}] \\ &\leq \alpha^{-1} 2^{n+1} \lambda \mu\{x \in [x_n, x_{n+1}]; w(x) > 2^n \beta \lambda\} . \end{aligned}$$

Let us put  $x_{j,k} := x_k$ .

We continue as in [7], that is, we sum in  $j$ , multiply both sides by  $\lambda^{\delta-1}$  and integrate from  $\lambda_0$  to  $\infty$  to get

$$\begin{aligned} &\int_{\lambda_0}^{\infty} \lambda^{\delta-1} \cdot \nu(\{x \in (a, b]; w(x) > \lambda\}) d\lambda \\ &\leq \sum_j [\alpha^{-1} 2^{n(j)} \int_0^{\infty} \lambda^{\delta} \mu\{x \in \{a_j, x_{j,n(j)}\}; w(x) > 2^{n(j)-1} \beta \lambda\} d\lambda \\ &+ \sum_{n \in \Gamma(j)} \alpha^{-1} 2^{n+1} \int_0^{\infty} \lambda^{\delta} \mu\{x \in [x_{j,n}, x_{j,n+1}]; w(x) > 2^n \beta \lambda\} d\lambda ] . \end{aligned}$$

Changing the variables  $\lambda \rightarrow 2^{n(j)-1} \beta \lambda$  and  $\lambda \rightarrow 2^n \beta \lambda$  on the right we arrive at

$$\frac{1}{\delta} \int_{(a,b]} w^{1+\delta}(x) d\mu - \frac{1}{\delta} \lambda_0^{\delta} \nu(a, b] \leq \frac{2}{\alpha \beta^{1+\delta} (\delta + 1)} \int_{(a,b]} w^{1+\delta}(x) d\mu ,$$

which yields (vi) if  $\delta$  is sufficiently small.

(vi) $\Rightarrow$ (vii) is obvious.

(vii) $\Rightarrow$ (viii): Let  $a < b < c$  be such that  $\mu(a, b] > 0$ . By (vii), for  $x \in [b, c)$ ,

$$\begin{aligned} \left( \frac{1}{\nu(a, c)} \int_{(a, b]} w^\delta d\nu \right)^{1/\delta} &\leq (M_\nu (w^\delta \chi_{(a, x]})(x))^{1/\delta} \\ &\leq K M_\mu (w \chi_{(a, c)})(x). \end{aligned}$$

As  $M_\mu$  is of weak type (1,1) with respect to  $\mu$  (see [25]), we have

$$\mu[b, c) \leq K [\nu(a, c)]^{1+1/\delta} \left[ \int_{(a, b]} w^\delta d\nu \right]^{-1/\delta},$$

or,  $\mu \in A_p^-(\nu)$  with  $p = 1 + \frac{1}{\delta}$ .

Now, many other implications follow by symmetry: For example, (viii) $\Rightarrow$ (x) is just a symmetric version of (i) $\Rightarrow$ (ii) and so on. In this way we subsequently obtain (viii) $\Rightarrow$ (x) $\Rightarrow$ (xi) $\Rightarrow$ (xii) $\Rightarrow$ (xiii) $\Rightarrow$ (xiv) $\Rightarrow$ (xv) $\Rightarrow$ (i). What remains is to include the exponential type conditions (ix) and (xvi) into the chain. We shall do it proving (i) $\Rightarrow$ (ix) $\Rightarrow$ (v); the symmetric argument will then give (viii) $\Rightarrow$ (xvi) $\Rightarrow$ (xiii) and round off the entire proof.

(i) $\Rightarrow$ (ix): Since  $w \in A_p^+(\mu)$ ,

$$\frac{\nu(a, b]}{\mu(a, b]} \left( \frac{1}{\mu[b, c)} \int_{[b, c)} w^{1-p'} d\mu \right)^{p-1} \leq K$$

holds for all  $a < b < c$  such that  $\mu(a, b) \leq \frac{1}{2}\mu(a, c) \leq \mu(a, b]$ . By Jensen's inequality applied to the convex function  $\exp((p' - 1)x)$  we get

$$\exp \left( \frac{1}{\mu[b, c)} \int_{[b, c)} \log \frac{1}{w} d\mu \right) \leq \left( \frac{1}{\mu[b, c)} \int_{[b, c)} \left( \frac{1}{w} \right)^{p'-1} d\mu \right)^{p-1}$$

and (ix) follows.

(ix) $\Rightarrow$ (v): Let  $\{a, b\}$  and  $\lambda$  be as in (v). First, let  $\{a, b\} = (a, b)$ . We put  $x_0 = b$  and find  $x_{-1}$  in order that

$$\mu(a, x_{-1}) \leq \frac{1}{2}\mu(a, b) \leq \mu(a, x_{-1}] .$$

We know that  $\mu(a, x) > 0$  for  $x \in (a, b)$ . Hence, having  $x_0, x_{-1}, \dots, x_k$ , we can choose  $x_{k-1}$  so that

$$\mu(a, x_{k-1}) \leq \frac{1}{2}\mu(a, x_k) \leq \mu(a, x_{k-1}] .$$

Now, we can proceed as in [11] and [19].

We find  $\alpha_k$  such that

$$\int_{[x_k, x_{k+1})} \log\left(\frac{1}{\alpha_k w}\right) d\mu = 0 .$$

Now, if (ix) holds for  $w$ , it also holds for  $\alpha_k w$ , whence

$$\alpha_k \lambda \leq \frac{\alpha_k}{\mu(a, x_k]} \int_{(a, x_k]} d\nu \leq K ,$$

and thus, using the inequality  $\log(1+x) \leq x$ ,

$$\begin{aligned} \mu(\{x \in [x_k, x_{k+1}); w(x) \leq \beta\lambda\}) &\leq \frac{1}{\log(1+1/(\alpha_k\beta\lambda))} \int_{[x_k, x_{k+1})} \alpha_k w d\mu \\ &\leq \frac{K}{\lambda \log(1+1/K\beta)} \int_{[x_k, x_{k+1})} d\nu . \end{aligned}$$

Summing in  $k$  and using  $\frac{\nu(a,b)}{\mu(a,b)} \leq 2\lambda$ ,

$$\begin{aligned} \mu(\{x \in (a, b); w(x) \leq \beta\lambda\}) &\leq \frac{K}{\lambda \log(1+1/(K\beta))} \nu(a, b) \\ &\leq \frac{2K}{\log(1+1/(K\beta))} \mu(a, b) , \end{aligned}$$

and it suffices to take  $\beta$  small enough.

If  $\{a, b\} = [a, b)$ , we choose  $x_0 = b$  and  $x_k, k \in -\mathbf{N}$  so that

$$\mu[a, x_{k-1}) \leq \frac{1}{2}\mu[a, x_k) \leq \mu[a, x_{k-1}] ,$$

and stop in the case that  $a = x_{k-1}$ . Then we proceed as above.

**3.4. Theorem.** Let  $p \in (1, \infty)$  and let us denote  $\sigma = w^{1-p'}$  and  $ds = \sigma d\mu$ . Then the following statements are equivalent.

- (i)  $w \in A_{\infty}^+(\mu)$  and  $\sigma \in A_{\infty}^-(\mu)$ ;
- (ii)  $M_{\mu}^-(w\chi_{(a,b]})(b) \leq K (M_{\sigma}(\frac{1}{\sigma}\chi_{(a,b]})(b))^{p-1}$  for all  $a < b$ ;
- (iii)  $w \in A_p^+(\mu)$ .

*P r o o f.* (i) $\Rightarrow$ (ii): Let  $a < b$  be such that  $\mu(a, b] > 0$ . We take  $x, y \in (a, b]$  such that

$$(3.4) \quad \mu(a, y) \leq \frac{1}{2}\mu(a, b] \leq \mu(a, y),$$

and

$$(3.5) \quad \mu(a, x) \leq \frac{1}{2}\mu(a, y] \leq \mu(a, x).$$

It is easy to see that  $a < x \leq y \leq b$ , and further

$$(3.6) \quad \mu[x, b] \leq 3\mu[x, y],$$

and

$$(3.7) \quad \mu[x, b] \leq 2\mu[y, b],$$

By (i),  $w \in A_r^+(\mu)$  for some  $r > 1$ . Jensen's inequality and (3.5) yield

$$\frac{1}{\mu(a, x]} \int_{(a, x]} w d\mu \cdot \exp \left( \frac{1}{\mu[x, y]} \int_{[x, y]} \log \frac{1}{w} d\mu \right) \leq K.$$

Similarly,  $\sigma \in A_q^-(\mu)$  for some  $q > 1$ . Jensen's inequality and (3.6), (3.7) yield

$$\frac{1}{\mu[y, b]} \int_{[y, b]} \sigma d\mu \cdot \exp \left( \frac{1}{\mu[x, y]} \int_{[x, y]} \log \frac{1}{\sigma} d\mu \right) \leq K.$$

Raising this inequality to  $p - 1$  and multiplying both inequalities we get

$$\left( \frac{1}{\mu(a, x]} \int_{(a, x]} w d\mu \right) \left( \frac{1}{\mu[y, b]} \int_{[y, b]} \sigma d\mu \right)^{p-1} \leq K,$$

which implies

$$\int_{(a,x]} w d\mu \leq K \mu(a,x] \left( M_s \left( \frac{1}{\sigma} \chi_{(a,b]} \right) (b) \right)^{p-1}.$$

Let us put  $x_0 = a$ ,  $x_1 = x$ , then

$$\int_{(x_0,x_1]} w d\mu \leq K \mu(x_0,x_1] \left( M_s \left( \frac{1}{\sigma} \chi_{(a,b]} \right) (b) \right)^{p-1}.$$

Suppose that  $x_0, \dots, x_k$  are chosen. If  $\mu(x_k, b] = 0$ , we we put  $x_{k+1} = b$  and stop. Otherwise we choose  $x_{k+1}$  as above with  $(a, b]$  replaced by  $(x_k, b]$ . Then  $a < x_1 < x_2 < \dots \leq b$ , and

$$(3.8) \quad \int_{(x_{k-1},x_k]} w d\mu \leq K \mu(x_{k-1},x_k] \left( M_s \left( \frac{1}{\sigma} \chi_{(a,b]} \right) (b) \right)^{p-1}.$$

It is easy to see that

$$\sum_k \mu(x_{k-1}, x_k] = \mu(a, b]$$

and

$$\sum_k \int_{(x_{k-1},x_k]} w d\mu = \int_{(a,b]} w d\mu.$$

Indeed, by (3.4) and (3.5) we have

$$\mu(x_1, b] = \mu(a, b] - \mu(a, x_1] \leq \frac{3}{4} \mu(a, b].$$

By the induction we get

$$\mu(x_k, b] \leq \left( \frac{3}{4} \right)^k \mu(a, b],$$

and so  $\lim_{k \rightarrow \infty} \mu(x_k, b] = 0$ .

Summing (3.8) in  $k$  we obtain

$$\frac{1}{\mu(a,b]} \int_{(a,b]} w d\mu \leq K \left[ M_s \left( \frac{1}{\sigma} \chi_{(a,b]} \right) (b) \right]^{p-1},$$

which implies (ii).

(ii) $\Rightarrow$ (iii): Let  $a < b < c$ . For  $x \in [b, c)$  we have

$$\frac{1}{\mu(a, c)} \int_{(a, b]} w d\mu \leq K \left( M_s \left( \frac{1}{\sigma} \chi_{(a, c)} \right) (x) \right)^{p-1}.$$

Then, since  $M_s$  is of weak type (1,1) with respect to  $ds$ , we obtain

$$\int_{[b, c)} w^{1-p'} d\mu \leq K \left( \frac{\mu(a, c)}{\int_{(a, b]} w d\mu} \right)^{p'-1} \mu(a, c),$$

which is (iii). For details see [19].

(iii) $\Rightarrow$ (i) follows easily from the fact that  $w \in A_p^+(\mu)$  is equivalent to  $\sigma \in A_{p'}^-(\mu)$  via Theorem 2.2.

**3.5. Remark.** It follows from Theorems 2.4 and 3.4 that for  $p \in (1, \infty)$

$$A_1^+(\mu) \cdot [A_1^-(\mu)]^{1-p} = A_p^+(\mu) = A_\infty^+(\mu) \cap [A_\infty^-(\mu)]^{1-p}.$$

#### 4. TWO SIDED PROBLEMS

In this section we assume merely that  $\mu$  and  $\nu$  are nonnegative Borel measures in  $\mathbf{R}$ , finite on compact sets. We shall use  $I$  for an interval in  $\mathbf{R}$ .

**4.1 Definition.** We say that  $\mu$  is comparable to  $\nu$ ,  $\mu \sim \nu$ , if there exist  $\alpha, \beta \in (0, 1)$  such that for  $E \subset I$

$$\nu(E) < \beta\nu(I) \quad \text{implies} \quad \mu(E) < \alpha\mu(I).$$

**4.2. Remark.** The relation  $\sim$  is equivalence, especially  $\mu \sim \nu$  if, and only if,  $\nu \sim \mu$ . Moreover, if  $\mu \sim \nu$ , then obviously  $\mu$  and  $\nu$  are mutually absolutely continuous. Thus, there exists  $w$  such that  $d\nu = w d\mu$ ,  $0 < w(x) < \infty$  almost everywhere.

**4.3. Definition.** We say that  $\nu \in A_\infty(\mu)$  if there exist positive  $\delta, K$  such that for all  $I$  and  $E \subset I$  we have

$$\frac{\mu(E)}{\mu(I)} \leq K \left( \frac{\nu(E)}{\nu(I)} \right)^\delta .$$

The following two theorems generalize (in  $\mathbf{R}^1$ ) the corresponding results from [7], [10], [14], [13] and [11].

**4.4. Theorem.** The following statements are equivalent.

- (i)  $\nu \sim \mu$ ;
- (ii)  $\nu \in A_\infty(\mu)$ ;
- (iii)  $\mu \in A_\infty(\nu)$ ;
- (iv) there exists  $p > 1$  such that  $w \in A_p(\mu)$ ;
- (v) there exists  $p > 1$  such that  $\frac{1}{w} \in A_p(\nu)$ ;
- (vi) there exist  $K, \delta > 0$  depending only on  $\mu, \nu$  such that for all  $I$

$$\left( \frac{1}{\mu(I)} \int_I w^{1+\delta} d\mu \right)^{1/(1+\delta)} \leq K \frac{1}{\mu(I)} \int_I w d\mu ;$$

- (vii) there exist  $K, \delta > 0$  depending only on  $\mu, \nu$  such that for all  $I$

$$\left( \frac{1}{\nu(I)} \int_I \left( \frac{1}{w} \right)^{1+\delta} d\nu \right)^{1/(1+\delta)} \leq K \frac{1}{\nu(I)} \int_I \frac{1}{w} d\nu ;$$

- (viii) there exists a constant,  $K$ , such that

$$\sup_I \left( \frac{1}{\mu(I)} \int_I w d\mu \right) \exp \left( \frac{1}{\mu(I)} \int_I \log \left( \frac{1}{w} \right) d\mu \right) \leq K ;$$

- (ix) there exists a constant,  $K$ , such that

$$\sup_I \frac{1}{\nu(I)} \int_I \log^+ \left( w(x) \frac{\mu(I)}{\nu(I)} \right) w(x) d\mu(x) \leq K .$$

*P r o o f.* We know from the one-sided theorems that  $A_\infty = \bigcup_{p>1} A_p$ . Thus, the equivalence of the first five statements easily follows from the fact that  $A_\infty = A_\infty^+ \cap A_\infty^-$ .

(v) $\Rightarrow$ (vi): Let  $\frac{1}{w} \in A_p(\nu)$  for some  $p$ . Then

$$\left( \int_I \left( \frac{1}{w} \right) w \, d\mu \right)^{1/p} \left( \int_I \left( \frac{1}{w} \right)^{1-p'} w \, d\mu \right)^{1/p'} \leq K \int_I w \, d\mu ,$$

that is,

$$\left( \frac{1}{\mu(I)} \int_I w^{p'} \, d\mu \right)^{1/p'} \leq K \frac{1}{\mu(I)} \int_I w \, d\mu ,$$

and (vi) with  $\delta = p' - 1$  follows.

(vi) $\Rightarrow$ (iii): By Hölder's inequality and the reverse Hölder inequality (vi),

$$\begin{aligned} \nu(E) &\leq \left( \int_E w^{1+\delta} \, d\mu \right)^{1/(1+\delta)} \cdot \mu(E)^{\delta/(1+\delta)} \\ &= \left( \frac{1}{\mu(I)} \int_E w^{1+\delta} \, d\mu \right)^{1/(1+\delta)} \cdot \mu(E)^{\delta/(1+\delta)} \cdot \mu(I)^{1/(1+\delta)} \\ &\leq \frac{K}{\mu(I)} \cdot \nu(I) \cdot \mu(I)^{1/(1+\delta)} \cdot \mu(E)^{\delta/(1+\delta)} = K \cdot \nu(I) \cdot \left( \frac{\mu(E)}{\mu(I)} \right)^{\delta/(1+\delta)} , \end{aligned}$$

which is (iii).

(ii) $\Leftrightarrow$ (viii) follows by using Theorem 3.3 and Remark 2.3.

Analogously we prove that (iii) is equivalent to

$$(4.1) \quad \sup_I \left( \frac{1}{\nu(I)} \int_I d\mu \right) \exp \left( \frac{1}{\nu(I)} \int_I \log w \cdot w \, d\mu \right) \leq K .$$

However, this is nothing but (ix). To see this it suffices to apply  $\log$  to both sides of (4.1) and use the argument from [13] and [14] to resume the “+” sign.

The rest of the theorem follows by symmetry.

The implications (vi) $\Rightarrow$ (ii) $\Rightarrow$ (i) and (iv) $\Rightarrow$ (i) are true also in  $\mathbf{R}^n$ ,  $n > 1$  (see [26], Chap. I, Lemma 12).

As currently adopted, we shall write  $w \in RH_{(1+\delta)}(\mu)$  if the statement (vi) from Theorem 4.4 is satisfied.

Similarly as in [27] (or [15]) it can be proved that

$$(4.2) \quad w \in RH_p(\mu) \quad \Leftrightarrow \quad w^p \in A_\infty(\mu) .$$

We shall use this result to prove the analog of Gehring's lemma (cf. [12]).

**4.5. Theorem.** Let  $I$  be an interval in  $\mathbf{R}^1$ ,  $g$  a nonnegative function on  $I$ ,  $g \in L_{q,\mu}(I)$ , and let  $q \in (1, \infty)$ . Assume that

$$(4.3) \quad \frac{1}{\mu(I')} \int_{I'} g^q d\mu \leq K \left( \frac{1}{\mu(I')} \int_{I'} g d\mu \right)^q$$

for all intervals  $I' \subset I$ . Then there exists  $\delta > 0$  such that for all  $p \in [q, q + \delta)$  we have

$$(4.4) \quad \frac{1}{\mu(I')} \int_{I'} g^p d\mu \leq K \left( \frac{1}{\mu(I')} \int_{I'} g^q d\mu \right)^{p/q}.$$

*P r o o f.* For the purpose of this theorem assume that  $\text{supp } \mu \subset I$ . It follows from (4.3) that  $g \in RH_q(\mu)$ , whence, by (4.2),  $g^q \in A_\infty(\mu)$ . In other words, there is a positive  $\varepsilon_0$  such that for any  $\varepsilon \in (0, \varepsilon_0)$  and all intervals  $J \subset \mathbf{R}$

$$\left( \frac{1}{\mu(J)} \int_J g^{q(1+\varepsilon)} d\mu \right)^{1/(1+\varepsilon)} \leq K \frac{1}{\mu(J)} \int_J g^q d\mu,$$

which is (4.4) with  $p = q(1 + \varepsilon)$  and  $I' = J \cap I$ .

**4.6. Theorem.** Let  $p \in (1, \infty)$ . The following statements are equivalent.

- (i)  $w \in A_\infty(\mu)$  and  $\sigma = w^{1-p'} \in A_\infty(\mu)$ ;
- (ii)  $w \in A_p(\mu)$ .

*P r o o f.* This equivalence follows easily from Theorem 3.4 and Remark 2.3. An alternative proof can be given employing Theorem 4.4. For details cf. [11], Theorem IV.2.17.

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