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# Cantor Type Sets of Positive Measure and Lipschitz Mappings*

**Introduction.** During the XVth Summer Symposium on Real Analysis A.M.Bruckner and J.Smítal asked the author the following question. If $E_1, E_2 \subset \mathbf{R}$ are non-empty, nowhere dense, perfect and each portion of $E_1$, or of $E_2$ is of positive Lebesgue measure does there exist a "nice" function, $f$, mapping $E_1$ onto $E_2$. By nice functions they meant $C^1$, $C^2$, differentiable, or Lipschitz functions. During the conference the author made an example of sets $E_1, E_2 \subset [0,1]$ satisfying the above conditions such that there exists no Lipschitz (and a fortiori no $C^1$ or $C^2$) function defined on $[0,1]$ which maps $E_1$ onto $E_2$. This paper contains this example. The differentiable case seems to be more involved. In fact we do not know whether there exists a differentiable $f : [0,1] \to \mathbf{R}$ satisfying $f(E_1) = E_2$ for the sets $E_1$ and $E_2$ defined in this paper.

This paper is organized in the following way. First we state our main result. Then we define the sets $E_1$ and $E_2$ and give a list of a few basic properties of these sets. Finally we prove a lemma which implies our main result by showing that any Lipschitz function mapping $E_1$ into $E_2$ is mapping $E_1$ onto a set of measure zero. On the other hand in the Lemma we also show that if $f$ is differentiable and Lipschitz at the points of a portion of $E_1$ then $f'(x) = 0$ for any $x$ belonging to this portion. If $f$ is differentiable on $[0,1]$ then $f'$ is Baire-one and it is easy to find portions of $E_1$ where $f$ is Lipschitz and differentiable. This shows that the example of a differentiable $f$, mapping $E_1$ onto $E_2$ should have zero derivative on a set which is dense and open in $E_1$. However this property (and some other properties) turned out to be insufficient (at least for the author) to verify the nonexistence of a differentiable $f$ mapping $E_1$ onto $E_2$.

In this paper we denote the Lebesgue measure of the set $A$ by $\lambda(A)$. We say that $f$ is Lipschitz with constant $L$ on a set $A$ if $|f(x) - f(y)| < L \cdot |x - y|$ for every $x, y \in A$.

**The statement of our main result.**

**Theorem.** *There exist non-empty, nowhere dense, perfect sets* $E_1, E_2 \subset [0,1]$ *such that each of their portions is of positive Lebesgue measure and there exists no Lipschitz function* $f : [0,1] \to \mathbf{R}$ *satisfying* $f(E_1) = E_2$.

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**Definition of the sets $E_1$ and $E_2$.** Put $N_0 = 1$. If $N_{n-1}$ is defined for an $n \in \{1,2,...\}$ put $M_n = 4^{2n}N_{n-1}$, $M'_n = 4^n M_n$, $N_n = 2 \cdot 4^n \cdot n \cdot M_n$, $N'_n = 4^n N_n$. Therefore $M_n, N_n, M'_n, N'_n$ are defined by induction for $n = 1,2,....$ Put $E_{1,0} = E_{2,0} = [0,1]$. If $E_{1,n-1}$, $E_{2,n-1}$ are defined for an $n \in \{1,2,...\}$ put

$$E_{1,n} = E_{1,n-1} \setminus \cup_{k=0}^{M_n-1}\left(\frac{k}{M_n} + \frac{2\cdot 4^{n-1}}{M'_n}, \frac{k}{M_n} + \frac{2\cdot 4^{n-1}+1}{M'_n}\right)$$

and

$$E_{2,n} = E_{2,n-1} \setminus \cup_{k=0}^{N_n-1}\left(\frac{k}{N_n} + \frac{2\cdot 4^{n-1}}{N'_n}, \frac{k}{N_n} + \frac{2\cdot 4^{n-1}+1}{N'_n}\right).$$

Finally let $E_1 = \cap_{n=1}^{\infty} E_{1,n}$ and $E_2 = \cap_{n=1}^{\infty} E_{2,n}$.

**Properties of the sets $E_1$ and $E_2$.** Some of the properties are obvious consequences of the definition of the sets $E_1$ and $E_2$ in these cases we leave the details of their verification to the reader.

**Property 1.** The sets $E_1$ and $E_2$ are non-empty, nowhere dense and perfect.

**Property 2.** The intervals which are contiguous to $E_1$ (to $E_2$) and are of length $1/M'_n$, (of length $1/N'_n$ ) are exactly the ones which are contiguous to $E_{1,m}$ (to $E_{2,m}$) for $m \geq n$ and are of length $1/M'_n$, (of length $1/N'_n$).

**Property 3.** If an interval of the form $[k/M_n, (k+1)/M_n] \subset E_{1,n-1}$ then at the $n$th step an interval of length $1/M'_n = 4^{-n}/M_n$ is removed from it. At the $m$th step $(m > n)$ an easy calculation shows that

$$\lambda\Big(\Big[\frac{k}{M_n}, \frac{k+1}{M_n}\Big] \cap \cup_{\ell=0}^{M_m-1}\Big(\frac{\ell}{M_m} + \frac{2\cdot 4^{m-1}}{M'_m}, \frac{\ell}{M_m} + \frac{2\cdot 4^{m-1}+1}{M'_m}\Big)\Big) = \frac{4^{-m}}{M_n}.$$

Therefore

$$\lambda(E_{1,m} \cap [k/M_n, (k+1)/M_n]) \geq (1 - 4^{-n} - 4^{-n-1} - ... - 4^{-m})/M_n$$

and hence $\lambda(E_1 \cap [k/M_n, (k+1)/M_n]) > 0$. This implies that any portion of $E_1$ is of positive Lebesgue measure.

**Property 4.** Similarly to Property 3 one can show that any portion of $E_2$ is of positive Lebesgue measure.

**Property 5.** If $I = [a,b] \subset E_{1,n}$ then the longest subinterval of $I$ contiguous to $I \cap E_1$ is of length not more than $1/M'_{n+1}$.

**Property 6.** Denote by $\nu_n$ the number of connected components (i.e. maximal closed subintervals) of $E_{1,n}$. If we denote by $\nu'_n$ the number of intervals contiguous to $E_{1,n}$ from $1, 0 \in E_{1,n}$ it follows that $\nu'_n + 1 = \nu_n$. (We do not count the halflines $(-\infty, 0)$ and $(1,\infty)$ as contiguous subintervals.) It is also clear that $\nu'_n \leq M_n + M_{n-1} + ... + M_1 < n \cdot M_n$ if $n > 1$. Therefore $\nu_n \leq n \cdot M_n$ if $n > 1$.

**Property 7.** Denote by $\kappa_n$ the number of connected components of $E_{2,n}$. Observe that

$$E_{2,n} = \Big([0,1] \setminus \cup_{k=0}^{N_n-1}\big(\frac{k}{N_n} + \frac{2 \cdot 4^{n-1}}{N'_n}, \frac{k}{N_n} + \frac{2 \cdot 4^{n-1}+1}{N'_n}\big)\Big)\setminus$$

$$\setminus \cup_{m=1}^{n-1} \Big(\cup_{k=0}^{N_m-1}\big(\frac{k}{N_m} + \frac{2 \cdot 4^{m-1}}{N'_m}, \frac{k}{N_m} + \frac{2 \cdot 4^{m-1}+1}{N'_m}\big)\Big) = A \setminus \cup_{m=1}^{n-1} B_m.$$

An easy calculation shows that the number of the connected components of $A \setminus B_1$ is more than $N_n - 4^{-1}N_n$, and in general the number of connected components of $A \setminus \cup_{m=1}^{j} B_m$ is more than $N_n - (4^{-1} + ... + 4^{-j})N_n$. Therefore $\kappa_n > N_n(1 - 4^{-1} - ... - 4^{-n+1}) > N_n(1 - \frac{1}{4 \cdot (1-(1/4))}) > N_n/2$.

**Property 8.** One can also easily verify that the length of the connected components of $E_{2,n}$ is less than $4^n/N'_n = 1/N_n$. We also remark that if $x$ and $y$ belong to different connected components of $E_{2,n}$ then the length of the largest subinterval of $[x,y]$ contiguous to $E_{2,n} \cap [x,y]$ is at least $1/N'_n$.

**Property 9.** Also a similar calculation shows that the length of the connected components of $E_{1,n}$ is more than $(2 \cdot 4^{n-1} - 1)/M'_n > 4^{n-1}/M'_n$ and less than $4^n/M'_n$. Using Property 2 it is easy to see that the endpoints of the connected components of $E_{1,n}$ all survive the inductive definition of $E_1$, that is, if $[c,d]$ is a connected component of $E_{1,n}$ then $c, d \in E_1$.

**Lemma.** *Assume that $E_1, E_2$ are defined as above, $(a,b) \subset [0,1]$, the function $f : [0,1] \to \mathbf{R}$ is continuous, furthermore $f$ is Lipschitz with constant $L$ on $(a,b) \cap E_1$, and $f((a,b) \cap E_1) \subset E_2$. Then $\lambda(f((a,b) \cap E_1)) = 0$.*

*If, in addition, we also assume that $f$ is differentiable at the points of $(a,b) \cap E_1$, then $f'(x) = 0$ for every $x \in (a,b) \cap E_1$.*

**Proof of the Lemma.** Choose $n_0$ such that

$$(1) \qquad \frac{M'_{n+1}}{N'_n} = \frac{4^{n+1} \cdot 4^{2(n+1)} \cdot N_n}{4^n N_n} = 4^{2n+3} > L \quad \text{for} \quad n \geq n_0.$$

Assume that $I$ denotes a connected component of $E_{1,n}$ and $x, y \in I \cap E_1$. According to Property 5 the longest subinterval in $I$, contiguous to $I \cap E_1$, is of length not more than $1/M'_n$. Denote by $J$ the closed interval with endpoints $f(x)$ and $f(y)$. Since $f$ is Lipschitz on $I \cap E_1$ the length of the longest subinterval in $J \setminus f(I \cap E_1)$ is not more than $L/M'_{n+1}$. Since by (1) we have $L/M'_{n+1} < 1/N'_n$ for $n \geq n_0$ the second half of Property 8 implies that $f(x)$ and $f(y)$ belong to the same connected component of $E_{2,n}$.

Denote by $\mu_n$ the number of connected components of $(a,b) \cap E_{1,n}$. We proved that $f$ maps points belonging to the same connected component of $(a,b) \cap E_{1,n}$ onto points belonging to the same connected component of $E_{2,n}$. It is also obvious that $\mu_n \leq \nu_n$. By Property 6 we have $\nu_n \leq n \cdot M_n$ for $n > 1$. Therefore $\mu_n \leq n \cdot M_n$. On the other hand, by Property 8 the length of the connected components of $E_{2,n}$ is less than $1/N_n$. This implies that

$$\lambda(f((a,b) \cap E_1)) \leq \frac{\nu_n}{N_n} \leq \frac{n \cdot M_n}{N_n} = \frac{n \cdot M_n}{2 \cdot 4^n \cdot n \cdot M_n} = \frac{1}{2 \cdot 4^n}.$$

Since this estimate is valid for any $n > n_0$ we obtain that $\lambda(f((a,b) \cap E_1)) = 0$.

Assume now that $f$ is differentiable at the points of $(a,b) \cap E_1$ and $x \in (a,b) \cap E_1$. For any $n \in \{1,2,...\}$ denote by $I_n = [c_n, d_n]$ the connected component of $E_{1,n}$ for which $x \in I_n$. Choose $n_1 \geq n_0$ such that $I_n \subset (a,b)$ for $n > n_1$. Denote by $J_n$ the connected component of $E_{2,n}$ for which $f(x) \in J_n$. By Property 8 the length of $J_n$ is less than $1/N_n$. By Property 9, $\lambda(I_n) = d_n - c_n > 4^{n-1}/M'_n$ and $c_n, d_n \in E_1$. Since $x \in [c_n, d_n]$ either $x - c_n \geq 4^{n-1}/2M'_n$, or $d_n - x \geq 4^{n-1}/2M'_n$. Thus

$$\min\left(\left|\frac{f(x) - f(c_n)}{x - c_n}\right|, \left|\frac{f(x) - f(d_n)}{x - d_n}\right|\right) \leq \frac{1}{N_n} \cdot \frac{2M'_n}{4^{n-1}} = \frac{1}{4^{n-1}n.}$$

Therefore letting $n \to \infty$ we can find a sequence $y_n \in \{d_n, c_n\}$ such that $y_n \to x$, and

$$\lim_{n \to \infty} \left|\frac{f(x) - f(y_n)}{x - y_n}\right| = 0.$$

This implies $f'(x) = 0$ and concludes the proof of our lemma.

**Proof of the Theorem.** Properties 1, 3 and 4 imply that $E_1$ and $E_2$ satisfy the assumption of the theorem, and the first part of the Lemma applied with $(a,b) = (0,1)$ shows that for any map satisfying $f(E_1) \subset E_2$ we have $\lambda(f(E_1)) = 0$ and hence $f(E_1) \neq E_2$.

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