Real Analysis Exchange Vol. 17 (1991-92)

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PARAMETRIC SEMICONTINUITY IMPLIES CONTINUITY

Abstract. *Suppose $a \neq b$ are non-zero. We prove that if either $\liminf_{t\to 0-}(f(x+bt)-f(x+at)) \geq 0$ or $\liminf_{t\to 0+}(f(x+bt)-f(x+at)) \geq 0$ holds at each point of a set $E \subset \mathbf{R}$ which has the Baire property (is measurable), then f is continuous at each point of E apart from a first category set (a set of measure zero). We prove this via a generalization of the Uher-Thomson symmetric covering theorem.*

1. Introduction. Investigations into the symmetric behavior of functions began in the 20's (see [7]). Later several papers appeared which specifically investigated the nature of sets with symmetry properties [1], [2], [3], [12]. By introducing the notion of symmetric cover, B. S. Thomson [13], [14] proposed a unified approach to both of these topics.

A set $S \subset \mathbf{R}^2$ is a *full symmetric cover of* $\mathbf{R}$ if, for every $x \in \mathbf{R}$, there is $\delta(x) > 0$ such that $(x-t, x+t) \in S$ $(0 < t < \delta(x))$. We say that S partitions an interval $[x, y]$ if there are points $x = x_0 < x_1 < \ldots < x_n = y$ such that $(x_{i-1}, x_i) \in S$ $(i = 1, \ldots, n)$.

We note that there is also a connection between symmetric covers and integration theory which was already observed by McGrotty in 1962 [9]. Recently D. Preiss and B.S. Thomson [11] defined an integral which integrates every convergent Fourier series based on the approximate symmetric covering theorem by C. Freiling and D. Rinne [5], [6].
It was proved by D. Preiss and Thomson [10]† that for every full symmetric cover there is a countable exceptional set E such that for every $x, y \notin E$, S partitions $[x, y]$. C. Freiling showed [4] that E can be chosen nowhere dense (a simple proof of this can be found in [8]).

AMS Mathematics classification numbers 26A03, 26A15

† The authors express their enduring gratitude for the inspiring last paragraph of this paper.

In [15] and [16] J. Uher proves that symmetric differentiability implies differentiability almost everywhere and semicontinuity implies continuity almost everywhere. In doing so he implicitely employs a new covering theorem which later was made explicit by Thomson in [14]. New in Uher's technique was the use of directed chains to obtain estimates on function growth.

Uher's proof is rather complicated, and our first goal is to simplify his argument. At the same time, we give a generalization of the category version of his theorem which also shows that in the covering theorems of [4], [8], [10] each subinterval of a dense open set can be partitioned into 5 intervals from S.

In the next section we give our simplified proofs of Uher's theorems. In Section 3 we show that these results remain valid if the collection of symmetric intervals is replaced by the slightly more general collection of $(x+at, x+bt)$, where a, b are given non-zero real numbers.

These theorems could be applied to obtain continuity, monotonicity or differentiability results (see [13], [14]). In Section 4 we provide one such application, a continuity result, which generalizes Uher's theorem. We chose this application because a literal generalization of Uher's theorem is not true (see Theorem 4.2). The appropriate generalization states that lower parametric semicontinuity from either side implies continuity a.e. (Theorem 4.1).

2. The Uher-Thomson covering theorems. Let $S \subset \mathbf{R}^2$. We shall say that $x_0, \ldots, x_n$ is an $n-$element $S-$chain, if $(x_{i-1}, x_i) \in S$ for every $i = 1, \ldots, n$.

Theorem 2.1. *Let $E \subset \mathbf{R}$ have the Baire property, let $S \subset \mathbf{R}^2$ and suppose that for every $x \in E$ there is a $\delta(x) > 0$ such that either $(x-t, x+t) \in S$ for every $0 < t < \delta(x)$, or $(x-t, x+t) \in S$ for every $0 > t > -\delta(x)$. Then there is an open set G such that $E \setminus G$ is of first category and for every $[x, y] \subset G$ there is a 5-element $S-$chain connecting x and y.*

Proof. Let $\mathcal{I}$ denote the family of all open intervals I such that for every $x, y \in I$, $x < y$, there is a 5-element $S-$chain connecting x and y. We shall prove that if E is residual in an interval (a, b) then there is an $I \subset (a, b)$, $I \in \mathcal{I}$. From this the statement of the theorem easily follows by taking G as the union of a maximal disjoint subfamily of $\mathcal{I}$.

Let E be residual in (a, b) and let $E^+ = \{x \in E : (x-t, x+t) \in S \text{ if } t > 0 \text{ is small enough}\}$, and $E^- = \{x \in E : (x-t, x+t) \in S \text{ if } -t > 0 \text{ is small enough}\}$. Then $E = E^+ \cup E^-$ and hence either E^+ or E^- is of second category in (a, b). We shall suppose that E^+ is of second category in (a, b); the same proof applies for the other case. Let

$$E_n^+ = \{x \in E : \ (x-t, x+t) \in S \text{ for every } 0 < t < 1/n\},$$

$$E_n^- = \{x \in E : \ (x-t, x+t) \in S \text{ for every } 0 > t > -1/n\} \quad (n = 1, 2, \ldots).$$

Then $E^+ = \cup_{n=1}^{\infty} E_n^+$ and hence there is an n and an open subinterval $I_1 \subset (a,b)$ such that $|I_1| < 1/n$ and E_n^+ is of second category in every subinterval of I_1. Let I_2 be the interval concentric with I_1 and of length $|I_1|/10$; we shall prove that $I_2 \in \mathcal{I}$.

Let $x, y \in I_2$, $x < y$; we show that there is a 5-element $S-$chain from x to y. We may assume that $x = 0$. It is enough to prove that there is a subinterval $J \subset (0.9y, y)$ such that for every $z \in J$ there is a 4-element $S-$chain from x to z. Indeed, if this is true then we choose a $w \in (0.9y, y) \cap E_n^+$ such that $2w - y \in J$, we connect 0 and $2w - y$ by a 4-element $S-$chain and complete this chain by adding $(2w - y, y) \in S$.

Since E is residual and $E_n^+ + \frac{3}{4}y$ is of second category in $(0.9y, y)$, $E \cap (E_n^+ + \frac{3}{4}y)$ is also of second category in $(0.9y, y)$. Therefore, by

$$E \cap (E_n^+ + \frac{3}{4}y) = \bigcup_{m=1}^{\infty} ((E_m^+ \cup E_m^-) \cap (E_n^+ + \frac{3}{4}y))$$

it follows that there is an m and an $\epsilon \in \{+,-\}$ such that $E_m^\epsilon \cap (E_n^+ + \frac{3}{4}y)$ is dense in an open interval $J \subset (0.9y, y)$. Let $z \in J$ be arbitrary. We shall construct four points p, q, r, s such that the chain of reflections about the points p, q, r, s takes 0 to z.

Since E is residual and E_n^+ is of everywhere second category in I_1, $A = \left(\frac{1}{2}E + \frac{3}{4}y\right) \cap \left(\frac{1}{2}E_n^+ + \frac{z}{4} + \frac{3}{8}y\right)$ is dense in J. Therefore we can choose $t \in A \cap J$ such that $|t - z| < 1/m$ and $t < z$ if $\epsilon = +$ and $t > z$ if $\epsilon = -$. We put $q = 2t - \frac{3}{2}y$, $r = 2t - \frac{z}{2} - \frac{3}{4}y$; then $q \in E$, $r \in E_n^+$ and $0 < q < r < t$.

Since $E_m^\epsilon \cap (E_n^+ + \frac{3}{4}y)$ is also dense in J, we can choose $s \in E_m^\epsilon \cap (E_n^+ + \frac{3}{4}y) \cap J$ arbitrarily close to t. Let $p = s - \frac{3}{4}y$, then $p \in E_n^+$. If we choose s close enough to t then the following inequalities can be also satisfied: $0 < p < q$, $2q - 2p < r$, $|2p - q| < \delta(q)$, $2p < q$ if $q \in E^+$ and $2p > q$ if $q \in E^-$, $|s - z| < 1/m$, $s < z$ if $\epsilon = +$ and $s > z$ if $\epsilon = -$.

Then the consecutive reflections about the points p, q, r, s constitute a 4-element chain between 0 and z, since $2q - 2r = z - \frac{3}{2}y$ and $2s - 2p = \frac{3}{2}y$. Also, as $p, r \in E_n^+$, $q \in E$, and $s \in E_m^\epsilon$, this chain is an $S-$chain. This completes the proof.

Theorem 2.2. *Let $E \subset \mathbf{R}$ be measurable, let $S \subset \mathbf{R}^2$ and suppose that for every $x \in E$ there is a $\delta(x) > 0$ such that either $(x - t, x + t) \in S$ for every $0 < t < \delta(x)$, or $(x - t, x + t) \in S$ for every $0 > t > -\delta(x)$. Then for almost every $x \in E$ there is a neighbourhood U of x such that for every $y \in U$ there is a 5-element $S-$chain connecting x and y.*

Proof. The Lebesgue outer measure will be denoted by λ. Let $D(H)$ denote the set of outer density points of $H \subset \mathbf{R}$. Then $D(H)$ is measurable,

$\lambda(H \setminus D(H)) = 0$, $D(D(H)) = D(H)$ and $D(H) \cap D(K) = D(H \cap D(K))$ for every $H, K \subset \mathbf{R}$. Also, for every $a, b \in \mathbf{R}$ we have $D(aH + b) = aD(H) + b$, where $aH + b = \{ax + b : x \in H\}$.

Deleting a set of measure zero from E, we may assume that $D(E) = E$. Let E^+, E^-, E_n^+, E_n^- be defined as in the proof of the previous theorem. We shall prove that for every

$$x \in \bigcup_{n=1}^{\infty} (D(E_n^+) \cup D(E_n^-))$$

there is a neighborhood U of x satisfying the requirements of the theorem. Since $E = \cup_{n=1}^{\infty}(E_n^+ \cup E_n^-)$ and thus $\lambda(E \setminus \cup_{n=1}^{\infty}(D(E_n^+) \cup D(E_n^-))) = 0$, this will prove the theorem.

Let $x = 0$ be a point of full outer density of E_n^+. Then there is a $0 < \delta < 1/n$ such that, for every $|y| < \delta$, each of the sets $2E_n^+ - y$, $\frac{1}{2}E + \frac{3}{4}y$, $\frac{2}{3}E_n^+ + \frac{y}{2}$, and $E \cap (E_n^+ + \frac{3}{4}y)$ intersects the interval $(0.9y, y)$ in a set of outer measure greater than $0.09y$.

We shall prove that for every $y \in (0, \delta)$ there is a 5-element $S-$chain from 0 to y. (A similar argument shows that for every $y \in (-\delta, 0)$ there is a 5-element $S-$chain from y to 0. Also, if $0 \in D(E_n^-)$ then there are 5-element chains from 0 to the points of a small left neighborhood of 0 and from the points of a right neighborhood of 0 to 0.)

The proof is basically the same as that of the previous theorem; only the selection of the number t is slightly more complicated.

Let $y \in (0, \delta)$ be fixed, and let $B = \{z \in (0.9y, y) : (z, y) \in S\}$. Then $\lambda(B) > 0.09y$. Indeed, if $C = E_n^+ \cap (0.95y, y)$ then $B \supset 2C - y$ and thus $\lambda(B) \geq \lambda(2C - y) > 0.09y$.

Therefore it is enough to show that there is a measurable set $F \subset (0.9y, y)$ such that $\lambda(F) > 0.01y$ and for every $z \in F$ there is a 4-element $S-$chain from 0 to z. Indeed, if this is true, then $F \cap B \neq \emptyset$. Selecting a $z \in F \cap B$, we can connect 0 and z by a 4-element $S-$chain and adding $(z, y) \in S$ we obtain a 5-element $S-$chain from 0 to y.

Since

$$\lambda\Big(E \cap (E_n^+ + \frac{3}{4}y) \cap (0.9y, y)\Big) > 0.09y,$$

there is an m and an $\epsilon \in \{+, -\}$ such that

$$\lambda\Big(E_m^{\epsilon} \cap (E_n^+ + \frac{3}{4}y) \cap (0.9y, y)\Big) > 0.04y.$$

We shall prove that the measurable set

$$F = \Big(\frac{1}{2}E + \frac{3}{4}y\Big) \cap D\Big(\frac{2}{3}E_n^+ + \frac{y}{2}\Big) \cap D\Big(E_m^{\epsilon} \cap (E_n^+ + \frac{3}{4}y)\Big) \cap (0.9y, y)$$

satisfies the requirements. It follows from the choice of δ that $\lambda(F) > 0.02y$. Let $z \in F$ be arbitrary, we shall construct four points p, q, r, s such that the chain of reflections about the points p, q, r, s takes 0 to z.

We have $z \in D(\frac{2}{3}E_n^+ + \frac{y}{2})$ and hence an elementary computation shows $z \in D(\frac{1}{2}E_n^+ + \frac{z}{4} + \frac{3}{8}y)$. Let

$$A = \left(\frac{1}{2}E + \frac{3}{4}y\right) \cap \left(\frac{1}{2}E_n^+ + \frac{z}{4} + \frac{3}{8}y\right) \cap D\left(E_m^\epsilon \cap (E_n^+ + \frac{3}{4}y)\right) \cap (0.9y, y),$$

then $z \in D(A)$.

Therefore we can choose $t \in A$ and define q and r such that they satisfy the same properties as in the previous proof. Since

$$t \in D\left(E_m^\epsilon \cap (E_n^+ + \frac{3}{4}y)\right),$$

t is a bilateral limit point of $E_m^\epsilon \cap (E_n^+ + \frac{3}{4}y))$, and we can choose s as before. The rest of the proof is the same.

Remark 2.3. The proofs of these theorems give slightly more than just the existence of 5-element chains: they also give the *direction* of the chains. In the proof of Theorem 2.1 the open set G is constructed in such a way that if I is a closed interval contained in G then for every $x, y \in I,\ x < y$, there is a 5-element $S-$chain connecting x and y *of the same direction* (independently of x and y). Namely, if E_n^+ (E_n^-) is of second category in every subinterval of I then the chains go from x to y (from y to x).

Similarly, in Theorem 2.2, the direction of chains connecting x and the points $y \in U$ is either from left to right (if $x \in D(E_n^+)$) or right to left (otherwise).

3. Parametric generalizations. In this section we generalize Theorems 2.1 and 2.2 by replacing the pairs $(x - t, x + t)$ by $(x + at, x + bt)$, where a and b are different and non-zero real numbers. These theorems are stated below. Because of the similarity between these proofs and those of the last section, we only prove 3.1.

Theorem 3.1. *Let a and b be different, non-zero real numbers. Let $E \subset \mathbf{R}$ have the Baire property, let $S \subset \mathbf{R}^2$ and suppose that for every $x \in E$ there is a $\delta(x) > 0$ such that either $(x + at, x + bt) \in S$ for every $0 < t < \delta(x)$, or $(x + at, x + bt) \in S$ for every $0 > t > -\delta(x)$. Then there is an open set G such that $E \setminus G$ is of first category and for every $[x, y] \subset G$ there is a 5-element $S-$chain connecting x and y. Moreover, for each component, I, of G the direction of the chains is the same. Namely, either*

(i) *for every* $[x,y] \subset I$, $x < y$, *there is a 5-element* S*—chain from* x *to* y, *or*
(ii) *for every* $[x,y] \subset I$, $x < y$, *there is a 5-element* S*—chain from* y *to* x.

Theorem 3.2. *Let* a *and* b *be different, non-zero real numbers. Let* $E \subset \mathbf{R}$ *be measurable, let* $S \subset \mathbf{R}^2$ *and suppose that for every* $x \in E$ *there is a* $\delta(x) > 0$ *such that either* $(x+at, x+bt) \in S$ *for every* $0 < t < \delta(x)$, *or* $(x+at, x+bt) \in S$ *for every* $0 > t > -\delta(x)$. *Then for almost every* $x \in E$ *there is a neighbourhood* U *of* x *such that for every* $y \in U$ *there is a 5-element* S*—chain connecting* x *and* y. *As for the direction of the chains, one of the following statements is true.*
(i) *For every* $y \in U$, $y < x$ *there is a 5-element* S*—chain from* y *to* x, *and for every* $y \in U$, $y > x$ *there is a 5-element* S*-chain from* x *to* y;
(ii) *For every* $y \in U$, $y < x$ *there is a 5-element* S*—chain from* x *to* y, *and for every* $y \in U$, $y > x$ *there is a 5-element* S*-chain from* y *to* x.

Proof of Theorem 3.1. Replacing the pair (a,b) by (da, db), where d is any non-zero real number, affects neither the condition nor the conclusion of the theorem. Therefore we may assume that $a = -1$.

Let $\mathcal{I}$ denote the family of all open intervals I satisfying at least one of conditions (i) and (ii) of Theorem 3.1. As in the proof of Theorem 2.1, it is enough to show that if E is residual in an interval I_0 then $\mathcal{I}$ contains a subinterval of I_0. We define E^+, E^-, E_n^+, E_n^- as in the proof of Theorem 2.1 (with the obvious modification), and suppose that E_n^+ is of second category in I_0. Then there is a subinterval $I_1 \subset I_0$ of length less than $1/n$ such that E_n^+ is of second category in every subinterval of I_1. Let the interval I_2 be concentric with I_1 and of length $\gamma|I_1|$, where γ is a positive constant depending only on b and is to be fixed later. We show that $I_2 \in \mathcal{I}$.

Let $x, y \in I_2$, $x < y$. We shall prove that if $-1 < b$ then there is a 5-element S-chain from x to y. (It can be proved similarly that if $-1 > b$ then there is a 5-element S-chain from y to x.)

We may assume $x = 0$. Let η be a small positive number to be fixed later. We shall prove that there is an interval $J \subset ((1-\eta)y, y)$ such that for every $z \in J$ there is a 4-element S—chain from $x = 0$ to z. Since E_n^+ is dense in I_1, we can choose a $w \in I_1 \cap E_n^+$ and a $0 < k < 1/n$ such that $w - k \in J$ and $w + bk = y$. Then we connect 0 and $z = w - k$ by a 4-element S—chain and complete this chain by adding $(w-k, w+bk) \in S$. This will finish the proof of the theorem.

We put $c = 1 - b^3/2(1+b)$. Since E is residual and E_n^+ is of everywhere second category in I_1, by choosing γ suitably, $E \cap (b^3 E_n^+ + cy)$ will be of second category in $((1-\eta)y, y)$. Therefore, by

$$E \cap (b^3 E_n^+ + cy) = \bigcup_{m=1}^{\infty} \left((E_m^+ \cup E_m^-) \cap (b^3 E_n^+ + cy) \right)$$

it follows that there is an m and an $\epsilon \in \{+,-\}$ such that $E_m^\epsilon \cap (b^3 E_n^+ + cy)$ is dense in an open interval $J \subset ((1-\eta)y, y)$. Let $z \in J$ be arbitrary. We shall construct four points $p, q, r, s \in I_1$ such that the chain of reflections about the points p, q, r, s takes 0 to z.

Since E is residual and E_n^+ is of everywhere second category in I_1,

$$A = \left(\frac{b^3}{1+b}E + cy\right) \cap \left(\frac{b^2}{1+b}E_n^+ + \frac{bz}{(1+b)^2} + \frac{cy}{1+b}\right)$$

is dense in J (again, supposing that γ is chosen small enough). Therefore we can choose $t \in A \cap J$ such that $|t-z| < |b|/m$ and $t < z$ if $\epsilon b > 0$ and $t > z$ if $\epsilon b < 0$. We put

$$q = \frac{1+b}{b^3}(t - cy),$$

and

$$r = \frac{1+b}{b^2}t - \frac{z}{(1+b)b} - \frac{cy}{b^2},$$

then we have $q \in E$ and $r \in E_n^+$.

Since $E_m^\epsilon \cap (b^3 E_n^+ + cy)$ is dense in J, we can choose $s \in E_m^\epsilon \cap (b^3 E_n^+ + cy) \cap J$ arbitrarily close to t. Let

$$p = \frac{1}{b^3}(s - cy),$$

then $p \in E_n^+$. We shall prove that s can be chosen in such a way that the reflections about the points p, q, r, s constitute an S-chain $x_0 = 0, x_1, x_2, x_3, x_4 = z$ between 0 and z. Then we have $x_1 = (1+b)p$, $x_2 = (1+b)(q - bp)$, $x_3 = (1+b)(r - bq + b^2 p)$. Since $p \in E_n^+$, $q \in E$, $r \in E_n^+$, $s \in E_m^\epsilon$, we have to show that with a suitably selected s the following inequalities are satisfied:

(1) $$0 < p < \frac{1}{n},$$

(2) $$0 < q - x_1 < \delta(q) \ \text{ if } \ q \in E^+ \ \text{ and } \ 0 > q - x_1 > -\delta(q) \ \text{ if } \ q \in E^-,$$

(3) $$0 < r - x_2 < \frac{1}{n},$$

(4) $$|s - x_3| < \frac{1}{m} \ \text{ and } \ \epsilon(s - x_3) > 0.$$

Since $q - x_1 = (1+b)b^{-3}(t-s)$, (2) is satisfied if s is close enough to t and is on the correct side of t. Since $s - x_3 = (z-s)/b$ and $|t-z| < |b|/m$, it follows from the choice of t that (4) holds if s is chosen close enough to t.

By selecting γ small enough, the construction gives $|p| < 1/n$ and $|r-x_2| < 1/n$. In order to show $0 < p$ and $0 < r - x_2$, we shall have to choose η small enough, according to the following computation.

Since $z, t, s \in J \subset ((1-\eta)y, y)$, we have

$$\left|p - \frac{y}{2(1+b)}\right| < \frac{\eta y}{|b|^3}, \quad \left|q - \frac{y}{2}\right| < \frac{1+b}{|b|^3}\eta y,$$

and

$$\left|r - \frac{2+b}{2(1+b)}y\right| < \left(\frac{1+b}{b^2} + \frac{1}{(1+b)|b|}\right)\eta y.$$

This implies, by $r - x_2 = r - (1+b)q + (1+b)bp$, that

$$\left|r - x_2 - \frac{y}{2(1+b)}\right| < K\eta y,$$

where K is a positive constant depending only on b. Therefore, choosing

$$0 < \eta < \min\left(\frac{|b|^3}{2(1+b)}, \frac{1}{2K(1+b)}\right),$$

$0 < p$ and $0 < r - x_2$ are satisfied, completing the proof.

4. Applications. Using the techniques described by Thomson in [13], [14] one can prove parametric versions of monotonicity, continuity, differentiability, etc. results. In this section we shall investigate one of these applications, a generalization of Uher's theorem ([16], Theorem 1, [14], Theorem 18 and Corollary 19). This theorem states that if $f : \mathbf{R} \to \mathbf{R}$ is upper or lower symmetrically semicontinuous at each point of a set E which has the Baire property (or is measurable), then f is continuous at nearly (almost) every point of E.

It is interesting to note that the literal generalization of Uher's theorem is false. Indeed, this would say that if a function $f : \mathbf{R} \to \mathbf{R}$ satisfies either

$$\limsup_{t\to 0-}(f(x+bt) - f(x+at)) \le 0 \tag{5}$$

or

$$\liminf_{t\to 0-}(f(x+bt) - f(x+at)) \ge 0 \tag{6}$$

at each point of a set E which has the Baire property (or is measurable), then f is continuous at nearly (almost) every point of E. As we shall see later (Theorem

4.2), this condition in general does not imply even the measurability of f, even if $E = \mathbf{R}$ and the conditions (5) and (6) are satisfied bilaterally.

However, for the symmetric case,

$$\begin{aligned}\limsup_{t\to 0-}(f(x+t)-f(x-t)) &= \limsup_{t\to 0+}(f(x-t)-f(x+t)) \\ &= -\liminf_{t\to 0+}(f(x+t)-f(x-t)),\end{aligned}$$

and thus the condition of upper symmetric semicontinuity could equally well be defined as

$$\liminf_{t\to 0+}(f(x+t)-f(x-t)) \geq 0.$$

Generalizing in this way to the parametric case does lead to the appropriate generalization of Uher's results.

A function f satisfying (5) will be called upper parametrically semicontinuous from the left at x. Lower parametric semicontinuity from the left is defined by (6). Parametric semicontinuity from the right is defined analogously.

Theorem 4.1. *Suppose a, b are different, non-zero real numbers, $E \subset \mathbf{R}$ has the Baire property (is measurable), and $f : \mathbf{R} \to \mathbf{R}$ is lower parametrically semicontinuous at each point of E either from the left or from the right. Then f is continuous at each point of E apart from a first category set (a set of measure zero).*

Proof. We shall prove that f is lower semicontinuous from one side and upper semicontinuous from the other side at nearly (almost) every point of E. This implies that f is continuous nearly (almost) everywhere on E as the following argument shows. Let E' be the set of points of E at which f is lower semicontinuous from one side and upper semicontinuous from the other side; then $E \setminus E'$ is of first category (is of measure zero). Let $\underline{f}(x) = \min(f(x), \liminf_{y\to x} f(y))$, $\overline{f}(x) = \max(f(x), \limsup_{y\to x} f(y))$, and put $H_{p,q} = \{x \in \mathbf{R} : \underline{f}(x) < p \text{ and } \overline{f}(x) > q\}$. It is easy to see that if $p < q$ then no bilateral point of accumulation of $H_{p,q}$ belongs to E'. This implies that $E' \cap H_{p,q}$ is countable for every $p < q$. Since f is continuous at each point of the set $E \setminus \cup\{H_{p,q} : p < q,\ p, q \text{ are rational}\}$, the assertion follows.

Let $\varepsilon > 0$ be fixed, and put $S_\varepsilon = \{(u, v) : f(v) > f(u) - \varepsilon\}$. If $x \in E$ and f is lower parametrically semicontinuous at x from the right, then $(x+at, x+bt) \in S_\varepsilon$ for every $0 < t < \delta(x)$. Similarly, if f is lower parametrically semicontinuous at x from the left, then $(x + at, x + bt) \in S_\varepsilon$ for every $0 > t > -\delta(x)$. Applying Theorem 3.1 (3.2), it is easy to see that for nearly (almost) every $x \in E$ there is a neighbourhood U of x such that either (i) or (ii) of Theorem 3.2 holds.

Obviously, if (i) holds then $f(y) > f(x) - 5\varepsilon$ for every $x < y,\ y \in U$ and $f(y) < f(x) + 5\varepsilon$ for every $x > y,\ y \in U$. Therefore,

$$\liminf_{y \to x+} f(y) \geq f(x) - 5\varepsilon \quad \text{and} \quad \limsup_{y \to x-} f(y) \leq f(x) + 5\varepsilon. \tag{7}$$

Similarly, if (ii) holds then

$$\limsup_{y \to x+} f(y) \leq f(x) + 5\varepsilon \quad \text{and} \quad \liminf_{y \to x-} f(y) \geq f(x) - 5\varepsilon. \tag{8}$$

This easily implies that at nearly (almost) every $x \in E$, either (7) or (8) holds for infinitely many $\varepsilon = 1/n$. In the first case f is lower semicontinuous from the right and upper semicontinuous from the left; in the second case f is lower semicontinuous from the left and upper semicontinuous from the right. This completes the proof.

Next we show that (5) and (6) do not imply continuity, even if they are interpreted bilaterally.

Theorem 4.2. *Let a and b be non-zero real numbers and let $\alpha = b/a$. Suppose that either α is transcendental, or it is algebraic and either α or one of its conjugates is real and positive. Then there is a non-measurable function $f : \mathbf{R} \to \mathbf{R}$ such that for every $x \in \mathbf{R}$ either*

$$f(x + bt) - f(x + at) \geq 0 \quad (t \in \mathbf{R}) \tag{9}$$

or

$$f(x + bt) - f(x + at) \leq 0 \quad (t \in \mathbf{R}) \tag{10}$$

holds.

Proof. Let $F = Q(\alpha)$ denote the smallest subfield of $\mathbf{R}$ containing α. If α is transcendental, then let $s = |\alpha|$, and if α is algebraic then let s be any of its positive real conjugates. In either case there is a field isomorphism $\phi : F \to \mathbf{R}$ such that $\phi(\alpha) = s > 0$.

Let U be a Hamel basis of $\mathbf{R}$ as a vector space over the field F. Then every $x \in \mathbf{R}$ has a unique representation of the form $x = \sum_{u \in U} c_u \cdot u$, where $c_u \in F$ for every u and $c_u = 0$ for all but a finite number of u's. Let an element $u_0 \in U$ be selected, and let $\gamma(x)$ denote the coefficient of u_0 in the expansion of x. Then the function $\gamma : \mathbf{R} \to F$ is additive, that is, it satisfies Cauchy's functional equation $\gamma(x+y) = \gamma(x) + \gamma(y)$. We put $\delta = \phi \circ \gamma$; then δ is also additive. Since the range of δ is $\phi(F) \neq \mathbf{R}$, it follows that δ is not linear. As it is well-known, this implies that δ is not measurable and the set $H = \{x : \delta(x) > 0\}$ is also non-measurable.

Let f denote the characteristic function of H. We shall prove that for every $x \in \mathbf{R}$ either (9) or (10) holds. If $c \in F$ and $y \in \mathbf{R}$ then $\gamma(cy) = c \cdot \gamma(y)$ by the definition of γ, and hence we have

$$\delta(cy) = \phi(\gamma(cy)) = \phi(c)\phi(\gamma(y)) = \phi(c)\delta(y).$$

This gives

$$\delta(x + bt) = \delta((1-\alpha)x + \alpha(x + at)) = \phi(1-\alpha)\delta(x) + \phi(\alpha)\delta(x + at) = \tag{11}$$

$$(1-s)\delta(x) + s\delta(x + at).$$

Suppose $0 < s < 1$. It follows from (11) that

1. $\delta(x + at) > 0$ implies $\delta(x + bt) > 0$ whenever $\delta(x) > 0$; that is, $x + at \in H$ implies $x + bt \in H$ if $x \in H$, and
2. $\delta(x + at) \leq 0$ implies $\delta(x + bt) \leq 0$ whenever $\delta(x) \leq 0$; that is, $x + at \notin H$ implies $x + bt \notin H$ if $x \notin H$.

Hence, f satisfies (9) whenever $x \in H$ and (10) otherwise.

If $s = \phi(\alpha) > 1$, then $\phi(1/\alpha) = \phi(a/b) \in (0, 1)$ and hence, interchanging a and b we obtain (9) and (10) according to whether $x \notin H$ or $x \in H$ holds. Finally, if $s = \phi(\alpha) = 1$ then necessarily $\alpha = 1$ and then both (9) and (10) are satisfied.

The previous theorem shows that Uher's theorem cannot be generalized using (5) and (6) in an unrestricted way. However, it leaves open the possibility that such a generalization exists for some pairs (a, b). In particular, if b/a is a negative rational number (as in Uher's case), there is some supporting evidence for this, as the following observation shows.

Remark 4.3. *Let $\alpha = b/a$ be a negative rational number, and suppose that $f : \mathbf{R} \to \mathbf{R}$ satisfies either* (9) *or* (10) *at every $x \in \mathbf{R}$. Then f is constant.*

Indeed, we may assume that a and b are coprime integers and $a > 0 > b$. If f satisfies (9) or (10) for every x then so does $c_1 f(c_2 x + c_3) + c_4$ for every $c_1, \ldots, c_4$. Therefore, if f is not constant then we may suppose that $f(0) = 0$ and $f(1) = 1$. Let $A = \{x : f(x) > 0\}$; then $1 \in A$ and $0 \notin A$. If x is such that (10) holds then $x + bt \in A$ implies $x + at \in A$. Setting $y = x + bt \in A$ this implies $x + \frac{a}{b}(y - x) \in A$ for every $y \in A$. Therefore we have

$$(1 - \frac{a}{b})x + \frac{a}{b}A \subset A. \tag{12}$$

Similarly, if x satisfies (9) then we obtain

$$(1 - \frac{b}{a})x + \frac{b}{a}A \subset A. \tag{13}$$

If $x \in \frac{a}{a-b}A$ then the left hand side of (12) contains 0. Since $0 \notin A$, this implies that (12) cannot hold and hence (13) is satisfied for every $x \in \frac{a}{a-b}A$. That is, we have

$$A + \frac{b}{a}A \subset A. \tag{14}$$

By considering $x \in \frac{b}{b-a}A$, a similar argument shows

$$A + \frac{a}{b}A \subset A. \tag{15}$$

It easily follows from (14) and (15) that

$$A + n \cdot \frac{a}{b}A + k \cdot \frac{b}{a}A \subset A$$

holds for every $n, k \in \mathbf{N}$. By $b < 0 < a$ this implies $A - naA + kbA \subset A$ for every $n, k \in \mathbf{N}$. Since a and $-b$ are positive and coprime integers, the set $\{na - kb : n, k \in \mathbf{N}\}$ contains every integer greater than $|ab|$. Therefore, by $(na - kb)A \subset naA - kbA$, we have $A - mA \subset A$ for every $m > |ab|$. Let $m > \max(|ab|, 2)$, then $(1 - m)A \subset A - mA \subset A$ and hence $(1 - m)^2 A \subset A$. Since $(1 - m)^2 \geq m > |ab|$, this implies $(1 - m)^2 A - (1 - m)^2 A \subset A$. Thus $0 \in A$, which is impossible.

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Received September 9, 1991