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Mean Value Properties for Symmetrically Differentiable Functions

Section 1. Introduction and Notation

Functions considered in this note will be real valued functions defined on the real line $\mathbb{R}$. Results obtained here will also apply to functions defined on intervals. Such a function $f : \mathbb{R} \mapsto \mathbb{R}$ is said to have a *symmetric derivative, $f^s(x)$, at the point x* if

$$f^s(x) = \lim_{h \to 0+} \frac{f(x+h) - f(x-h)}{2h}.$$

We say that f is *symmetrically differentiable* if $f^s(x)$ exists (infinite values permitted) for each $x \in \mathbb{R}$, and say that a symmetrically differentiable function f possesses the *Mean Value Property* (MVP) if for each $a < b$, there exists a $\xi \in (a, b)$, such that

$$f^s(\xi) = \frac{f(b) - f(a)}{b - a}.$$

One shortcoming of symmetric differentiation, when compared to ordinary differentiation, is that continuous symmetrically differentiable functions with finite symmetric derivatives need not possess the MVP, as evidenced by the absolute value function. The purpose of this note is to observe some conditions which may be placed on a symmetrically differentiable f under which it will, at least, possess a weakened form of the MVP, and then to seek additional requirements on f^s which will guarantee that f has the MVP. In particular, we shall show (Corollary 1) that a symmetrically differentiable, Baire 1, Darboux function f will possess the MVP if and only if f^s possesses what we shall call the weak Darboux property. We need to define a few more terms and state some background results.

Recall that we say a function $f : \mathbb{R} \mapsto \mathbb{R}$ is *Darboux* (or $f \in \mathcal{D}$) if for every $a < b$, and every C strictly between $f(a)$ and $f(b)$, there is a $c \in (a, b)$ such that $f(c) = C$. We shall say that a function f is *strongly Darboux* ($f \in \mathcal{D}^+$) if $f + l \in \mathcal{D}$ for every linear function $l(x) = cx$, $c \in \mathbb{R}$. Next, we shall say that a function f is *weakly Darboux* ($f \in \mathcal{D}^-$) if for every $a < b$, every C strictly between $f(a)$ and $f(b)$, and every $\delta > 0$, there is a $c \in (a - \delta, b + \delta)$ such that $f(c) = C$. Further, we shall say that f is *very weakly Darboux* ($f \in \mathcal{D}^{--}$) if for each $x \in \mathbb{R}$,

$$\liminf_{t \to x} f(t) \leq f(x) \leq \limsup_{t \to x} f(t).$$

We pause to make the following simple observation.

Remark 1 *The following strict inclusions hold:*

$$\mathcal{D}^+ \subset \mathcal{D} \subset \mathcal{D}^- \subset \mathcal{D}^{--}.$$

Proof. Certainly, the only inclusion that might require verification is the last. Assume that $f \in \mathcal{D}^-$ and suppose there is a point x where $f(x) < \liminf_{t \to x} f(t)$. Let $\{t_n\}$ be a sequence of points converging to x for which $\lim_{n \to \infty} f(t_n) = L \equiv \liminf_{t \to x} f(t)$. Without loss of generality we may assume that each t_n is greater than x. By the weak Darboux property we know that for each n there is a point $c_n \in (x - 1/n, t_n + 1/n)$ for which $f(c_n) = (f(x) + f(t_n))/2$. Since $\{c_n\}$ converges to x and $\{f(c_n)\}$ converges to $(f(x) + L)/2 < L$, we have reached a contradiction. Consequently, no such point x exists. Similarly, there can be no point x where $f(x) > \limsup_{t \to x} f(t)$, and hence $f \in \mathcal{D}^{--}$. Furthermore, the Baire 1 function g which assumes the value 0 on $(-\infty, 0]$ and 1 on $(0, \infty)$ clearly belongs to $\mathcal{D}^{--} \setminus \mathcal{D}^-$.

In [2] A. M. Bruckner gives an example of a Baire 2 function $f \in \mathcal{D}$ such that $f(x) + x \notin \mathcal{D}$ and hence $f \in \mathcal{D} \setminus \mathcal{D}^+$. Thus the first inclusion is strict.

Finally, the Baire 1 function g given by

$$g(x) = \begin{cases} 1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ \sin(1/x) & \text{if } x > 0 \end{cases}$$

clearly belongs to $\mathcal{D}^- \setminus \mathcal{D}$.

As in [5], we say that a function f belongs to class M_{-1} if it is measurable and very weakly Darboux. This class M_{-1} has previously found use in the study of symmetric differentiation, in particular in monotonicity results. In [10] C. E. Weil proved the following result, in which $\underline{f}^s(x)$ denotes the lower symmetric derivate of f at x; i.e.,

$$\underline{f}^s(x) = \liminf_{h \to 0+} \frac{f(x+h) - f(x-h)}{2h}.$$

Theorem W *Let $f : \mathbb{R} \mapsto \mathbb{R}$ be a Baire 1, Darboux function with $\underline{f}^s(x) \geq 0$ for all x. Then f is nondecreasing.*

Evans [5] extended this to the following:

Theorem E1 *Let $f : \mathbb{R} \mapsto \mathbb{R}$ be a function for which $\underline{f}^s(x) \geq 0$ for all x. Then f is nondecreasing if and only if $f \in M_{-1}$.*

A symmetrically differentiable function f will be said to possess the *Quasi Mean Value Property* (QMVP) if for each $a < b$, there exist ξ_1 and ξ_2 in (a, b), such that

$$f^s(\xi_1) \leq \frac{f(b) - f(a)}{b - a} \leq f^s(\xi_2),$$

and will be said to possess the *weak Quasi Mean Value Property* (wQMVP) if for each $a < b$, there exist ξ_1 and ξ_2 in $[a, b]$, such that

$$f^s(\xi_1) \leq \frac{f(b) - f(a)}{b - a} \leq f^s(\xi_2).$$

Section 2. Mean Value Results

We begin by stating the following two known results concerning mean value type properties for symmetrically differentiable functions. The first is due to C. E. Aull [1] and the second to Evans [5].

Theorem A *A continuous, symmetrically differentiable function possesses the QMVP.*

If no type of continuity condition is assumed, then we cannot expect to get the QMVP or even the wQMVP as the characteristic function of the origin demonstrates. We do have the following, however.

Theorem E2 *A symmetrically differentiable function in class M_{-1} possesses the wQMVP.*

Without additional hypotheses this latter result cannot be improved to obtain the QMVP. For example, consider the function which is 0 for $x \leq 0$ and 1 for $x > 0$, and let $a = 0$ and $b = 1$. However, if we assume that the function in Theorem E2 has the strong Darboux property, then we can recover the QMVP.

Theorem 1 *A strongly Darboux, symmetrically differentiable function possesses the QMVP.*

Proof. Let $a < b$. We need to show the existence of ξ_1 and ξ_2 in (a, b), such that

$$f^s(\xi_1) \leq \frac{f(b) - f(a)}{b - a} \leq f^s(\xi_2).$$

We shall establish the existence of ξ_1 here and ξ_2 can be handled analogously.

Suppose that no such point $\xi_1 \in (a, b)$ exists. Then the function

$$F(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a)$$

is symmetrically differentiable with $F^s(x) > 0$ for all $x \in (a, b)$, and $F(a) = F(b)$. This F is a Darboux function, being the sum of a strongly Darboux function, a linear function, and a constant function. Furthermore, F is measurable. This follows from the result of J. Uher [9] that any symmetrically differentiable function is measurable. (See also [8].) Theorem E1 may now be employed to conclude that F is nondecreasing on (a, b) and since $F \in \mathcal{D}$, it must also be nondecreasing on $[a, b]$. However, since $F(a) = F(b)$, this means that F is constant on $[a, b]$, contradicting the premise that $F^s(x) > 0$ for all $x \in (a, b)$. Hence, the point ξ_1 must exist and our proof is complete.

We now look for a necessary and sufficient condition to place on the symmetric derivative of the function f in the hypothesis of Theorem A or Theorem 1 in order that QMVP can replaced by MVP in the conclusion. It is fairly easy to see that requiring $f^s \in M_{-1}$ is not enough, while requiring $f^s \in \mathcal{D}$ is too strong. To see the former, notice that the function $f(x) = |x|$ has $f^s \in M_{-1}$, yet f fails to have the MVP. For the latter, we shall show the existence of a continuous function f which is finitely symmetrically differentiable, possesses the MVP, and yet has a symmetric derivative which fails to possess the Darboux property. To this end, let f be given by

$$f(x) = \begin{cases} -x & \text{if } x \in (-\infty, 0] \\ x^2 \sin(1/x^2) & \text{if } x \in (0, \infty). \end{cases}$$

This f is differentiable everywhere except $x = 0$ where it is symmetrically differentiable with symmetric derivative $f^s(0) = -1/2$. It is easy to see that this function has the MVP. Indeed, if the two points $a < b$ satisfy $ab \geq 0$, then we may apply the mean value theorem for ordinary derivatives; and if $a < 0 < b$, then there are infinitely many choices for points $c \in (0, b)$ such that

$$f^s(c) = f'(c) = 2c \sin(1/c^2) - \frac{2\cos(1/c^2)}{c} = \frac{f(b) - f(a)}{b - a}.$$

However, f^s clearly fails to have the Darboux property as $f^s(x) \equiv -1$ on $(-\infty, 0)$, yet $f^s(0) = -1/2$.

These observations, combined with Remark 1, make it seem plausible that the weak Darboux property might be the condition we seek and, indeed, we have the following theorem.

Theorem 2 *If the symmetrically differentiable function $f \in \mathcal{D}^+$, then f possesses the MVP if and only if $f^s \in \mathcal{D}^-$.*

Proof. Let $f \in \mathcal{D}^+$ be symmetrically differentiable, with $f^s \in \mathcal{D}^-$, and let $a < b$. From Theorem 1 we know that there are ξ_1 and ξ_2 in (a, b), such that

$$f^s(\xi_1) \leq \frac{f(b) - f(a)}{b - a} \leq f^s(\xi_2).$$

Without loss of generality we assume that $\xi_1 < \xi_2$. Choose δ such that $(\xi_1 - \delta, \xi_2 + \delta) \subset (a, b)$. Employing the weak Darboux property of f^s, we know there is a $\xi \in (\xi_1 - \delta, \xi_2 + \delta)$ such that

$$f^s(\xi) = \frac{f(b) - f(a)}{b - a}.$$

Hence f has the MVP.

Conversely, suppose that $f \in \mathcal{D}^+$ is a symmetrically differentiable function which possesses the MVP. Let $a < b$ and suppose that $f^s(a) < C < f^s(b)$. Define the function G by $G(x) = f(x) - Cx$. Then G is a Darboux, symmetrically differentiable function which has the MVP. Furthermore, $G^s(a) < 0$ and $G^s(b) > 0$. Let $\delta > 0$. Since $G^s(a)$ and $G^s(b)$ have opposite signs, G clearly fails to be monotone on $(a - \delta, b + \delta)$. This fact, together with the fact that G is a Darboux function, implies that there exist two points x_1 and x_2 in $(a - \delta, b + \delta)$ such that $G(x_1) = G(x_2)$. Applying the MVP, we obtain a point $c \in (x_1, x_2)$ such that $G^s(c) = 0$. Consequently, $f^s(c) = C$, and it follows that f^s possesses the weak Darboux property.

Corollary 1 *If f is a Baire 1, Darboux, symmetrically differentiable function, then f possesses the MVP if and only if $f^s \in \mathcal{D}^-$.*

Proof. If $f \in \mathcal{B}_1\mathcal{D}$, then $f \in \mathcal{D}^+$ as it is well known that the sum of a $\mathcal{B}_1\mathcal{D}$ function and a continuous function is $\mathcal{B}_1\mathcal{D}$; e.g., see [2].

Corollary 2 *If f is a Darboux, finitely symmetrically differentiable function, then f possesses the MVP if and only if $f^s \in \mathcal{D}^-$.*

Proof. Z. Charzyński [3] showed that if a function has upper symmetric derivate less than $+\infty$ everywhere, then it is continuous at every point with the exception of a scattered set, where a scattered set is a set having no dense in itself subset. (See also [6].) Such a function must be a Baire 1 function and we may now apply Corollary 1.

In light of the fact that a symmetrically differentiable function is differentiable almost everywhere [9], it might be anticipated that the point ξ of the MVP in Theorem 2 or Corollaries 1 and 2 can always be chosen to be a point of ordinary differentiability. However, this is not the case, even if f is continuous and $f^s \in \mathcal{D}$ as the following construction shows.

Remark 2 *There is a continuous, symmetrically differentiable function f with the following properties:*

1. $f^s \in \mathcal{D}$.

2. *f is even, i.e., $f(x) = f(-x)$ for every x, and hence $f^s(0) = 0$.*

3. *The ordinary derivative never assumes the value 0.*

Proof. For each natural number n let $I_n = [1/(n+1), 1/n]$. Let $g_{1,n}(x) = n(x - 1/(n+1)) + 1/(n+1)$, and $g_{2,n}(x) = (1/n)(x - 1/n) + 1/n$; i.e., the graph of $g_{1,n}$ is a straight line having slope n passing through the point $(1/(n+1), 1/(n+1))$ and the graph of $g_{2,n}$ is a straight line having slope $1/n$ passing through the point $(1/n, 1/n)$. Define the function g by

$$g(x) = \begin{cases} x & \text{if } x > 1 \\ \min\{g_{1,n}(x), g_{2,n}(x)\} & \text{if } x \in I_n \\ 0 & \text{if } x = 0 \\ g(-x) & \text{if } x < 0. \end{cases}$$

This function g is not differentiable at the origin, having -1 as a left derivate and 1 as a right derivate there. However, g is symmetrically differentiable. Clearly, $g^s(0) = 0$ and the only other places in $(0, \infty)$ where g fails to be differentiable are at the endpoints of each I_n and at the point $x_n \in I_n$ where $g_{1,n}(x_n) = g_{2,n}(x_n)$. At each of these points g has a finite left and right derivative and hence a symmetric derivative. For every $x > 0$, $g^s(x) > 0$ and in every open interval having 0 as a left endpoint g^s assumes arbitrarily large and arbitrarily small positive values. The only shortcoming of g is that $g^s \notin \mathcal{D}$. We clearly may remedy this situation by "rounding" the corners of the graph of g on each I_n, to obtain a differentiable function f on $(0, \infty)$ with $f'(x) > 0$ for all $x > 0$. We set $f(0) = 0$, and $f(x) = f(-x)$ for $x < 0$. Then f will have all the required properties.

Before concluding this paper, we wish to take note of the analogues of the prior results in the situation where symmetric differentiation is replaced by approximate symmetric differentiation. A function $f : \mathbb{R} \mapsto \mathbb{R}$ is said to have a *approximate symmetric derivative, $f^s_{ap}(x)$, at the point x* if

$$f^s_{ap}(x) = ap - lim_{h \to 0+} \frac{f(x+h) - f(x-h)}{2h}.$$

We say that f is *approximately symmetrically differentiable* if $f^s_{ap}(x)$ exists (infinite values permitted) for each $x \in \mathbb{R}$. H. Croft's [4] familiar example of a Baire 1, Darboux function which is zero almost everywhere, but not identically zero , provides an example showing that the direct analogue of Corollary 1 will not hold, as this function will clearly have approximate symmetric derivative zero everywhere. If, however, we strengthen the Baire 1, Darboux condition to approximate continuity, then the following two theorems result.

Theorem 3 *If f is an approximately continuous, approximately symmetrically differentiable function, then for each $a < b$, there exist ξ_1 and ξ_2 in (a, b), such that*

$$f^s_{ap}(\xi_1) \leq \frac{f(b) - f(a)}{b - a} \leq f^s_{ap}(\xi_2).$$

Proof. This result may be confirmed by following the proof given for Theorem 1 except that instead of employing the monotonicity Theorem E1, one should use the much deeper monotonicity result of C. Freiling and D. Rinne [7], which states that if an approximately continuous function has a non-negative lower approximate symmetric derivative everywhere, then the function is non-decreasing.

Having made this observation, the next result follows precisely in the same manner that Theorem 2 followed from Theorem 1.

Theorem 4 *If f is an approximately continuous, approximately symmetrically differentiable function, then the following are equivalent:*

1. $f^s_{ap} \in \mathcal{D}^-$.

2. *For each $a < b$, there exists a $\xi \in (a, b)$, such that*

$$f^s_{ap}(\xi) = \frac{f(b) - f(a)}{b - a}.$$

Corresponding to Remark 2, we wish to observe that the point ξ of the previous theorem need not be a point of either approximate differentiability or of symmetric differentiability.

Remark 3 *There is a continuous, approximately symmetrically differentiable function f with the following properties:*

1. $f^s_{ap} \in \mathcal{D}$.
2. *There is a set S of positive numbers having right density one at 0 such that* $f(x) = f(-x)$ *for all* $x \in S$*, and hence* $f^s_{ap}(0) = 0$.
3. *Neither the approximate derivative nor the symmetric derivative of f ever assumes the value* 0.

Proof. If S is a set, we shall let $-S = \{x : -x \in S\}$. For each natural number $n \geq 3$ let $J_n = \left[\frac{1}{2^{n+1}}, \frac{1}{2^n}\right] = [a_n, e_n]$. We divide J_n into two subintervals, $H_n = \left[\frac{1}{2^{n+1}}, \frac{1}{2^{n+1}} + \frac{1}{2^{2n}}\right] = [a_n, d_n]$ and $I_n = \left[\frac{1}{2^{n+1}} + \frac{1}{2^{2n}}, \frac{1}{2^n}\right] = [d_n, e_n]$. We set

$$r_n = \frac{|H_n|}{|J_n|} = \frac{1}{2^{n-1}},$$

$$s_n = \frac{|H_n|}{|I_n|} = \frac{1}{2^{n-1} - 1},$$

and

$$t_n = \frac{|J_n| - (1 + r_n)|H_n|}{r_n|H_n|} = 2^{2n-2} - 2^{n-1} - 1.$$

We label two interior points of H_n, each close to an endpoint as $b_n = a_n + r_n|H_n|$ and $c_n = d_n - r_n|H_n|$. Note that $a_n < b_n < c_n < d_n < e_n$.

We shall first define a function g. We set $g(x) = 1/8$ for all x with $|x| \geq 1/8$. We define g on each I_n by

$$g(x) = s_n(x - e_n) + e_n,$$

and on each $-I_n$ by

$$g(x) = g(-x).$$

Since $\bigcup_{n=3}^{\infty} I_n$ clearly has right density one at the origin, regardless how we define g on $\{0\} \cup \bigcup_{n=3}^{\infty} H_n$, $g^s_{ap}(0) = 0$.

On H_n we define g by

$$g(x) = \begin{cases} s_n(x - a_n) + a_n & \text{if } x \in [a_n, c_n], \\ t_n(x - d_n) + e_n - |H_n| & \text{if } x \in [c_n, d_n], \end{cases}$$

and on $-H_n$ by

$$g(x) = \begin{cases} -s_n(x + d_n) + e_n - |H_n| & \text{if } x \in [-d_n, -b_n], \\ -t_n(x + a_n) + a_n & \text{if } x \in [-b_n, -a_n]. \end{cases}$$

Finally, we set $g(0) = 0$. On each J_n and each $-J_n$, g is a continuous, piecewise linear function. On each open interval having the origin as a left endpoint, g^s is positive and takes on arbitrarily large and arbitrarily small positive values; and on every open interval having the origin as a right endpoint, g^s is negative and takes on arbitrarily large and small negative values. Since $g^s_{ap}(0) = 0$ and $g(x)/x \geq 1/2$ for every $x > 0$, it follows that g is not approximately differentiable at 0. To see that g is not symmetrically differentiable at the origin, it will suffice to show that

$$\lim_{n\to\infty} \frac{g(-c_n) - g(c_n)}{2c_n} \neq 0.$$

Toward this end, note that

$$c_n = \frac{1}{2^{n+1}} + \frac{1}{2^{2n}} - \frac{1}{2^{n-1}}\frac{1}{2^{2n}} = \frac{2^{2n-2} + 2^{n-1} - 1}{2^{3n-1}},$$

$$g(c_n) = \frac{1}{2^{n+1}} + \frac{1}{2^{3n-1}} = \frac{2^{2n-2} + 1}{2^{3n-1}},$$

and

$$g(-c_n) = \frac{1}{2^n} - \frac{1}{2^{2n}} - \frac{1}{2^{n-1} - 1}\frac{1}{2^{n-1}}\frac{1}{2^{2n}} > \frac{2^{2n-1} - 2^{n-1} - 1}{2^{3n-1}}.$$

Hence,

$$\frac{g(c_n) - g(-c_n)}{2c_n} < \frac{2^{2n-3} - 2^{2n-2} + 2^{n-2} + 1}{2^{2n-2} + 2^{n-1} - 1},$$

and since this latter fraction tends to $-\frac{1}{2}$ as $n \to \infty$, g is not symmetrically differentiable at the origin.

This function g has all the properties mentioned in the statement of this remark except g^s_{ap} fails to have the Darboux property. This, however, is easily remedied by rounding the corners of the graph of g on each J_n and $-J_n$ to obtain a function f which is differentiable on both $(-\infty, 0)$ and $(0, \infty)$. This function f will then have all of the stated properties.

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