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# APPROXIMATE SYMMETRIC DERIVATIVES ARE UNIFORMLY CLOSED

It is well-known that the class of all derivatives is closed with respect to the uniform convergence. It is also known that the class of symmetric derivatives and the class of approximate derivatives are uniformly closed, [K] and [W]. Consequently, the intersection of these two classes is obviously uniformly closed. In contrast to this observation, we shall prove that the class of approximate symmetric derivatives of Lebesgue measurable functions is also uniformly closed. We shall also show that there is an approximate symmetric derivative of a measurable function that is not simultaneously a symmetric derivative and an approximate derivative. Note that all these functions belong to the first Baire class [L].

Let us begin with some definitions. We shall deal with finite real functions defined on an open real interval $I_0$. For a measurable set $A \subset I_0$, the upper density of $A$ at $x$ is $\overline{d}(A,x) = \limsup \lambda(A \cap [x-h, x+h])/2h$ as $h \to 0+$ ($\lambda$ is the Lebesgue measure on the real line $R$). The lower density $\underline{d}$ is defined in a similar manner. When these two values are the same, the common value $d(A,x)$ is the density of $A$ at $x$. The upper symmetric derivative of $F$ at $x$ is $\overline{F}^s(x) = \limsup(F(x+h)-F(x-h))/2h$ as $h \to 0+$. The lower symmetric derivative of $F$ at $x$, $\underline{F}^s(x)$, is defined similarly. When these two coincide, the common value $F^s(x)$ is the symmetric derivative of $F$ at $x$. The lower approximate symmetric derivative of $F$ at $x$, $\underline{F}^s_{ap}(x)$, is the least upper bound of the collection of $\alpha$ such that the set $A(\alpha) = \{t : (F(x+t) - F(x-t))/2h < \alpha\}$ has density zero at 0. The upper approximate symmetric derivative, $\overline{F}^s_{ap}(x)$, is defined similarly. When $\overline{F}^s_{ap}(x) = \underline{F}^s_{ap}(x)$, the common value $F^s_{ap}(x)$ is the approximate symmetric derivative of $F$ at $x$.

In the next proof we will follow some ideas of [M] and [Br], pp. 154-157.

**Lemma.** *Let $F : I_0 \to R$ be a nondecreasing function. Then $\underline{F}^s(x_0) = \underline{F}^s_{ap}(x_0)$ and $\overline{F}^s(x_0) = \overline{F}^s_{ap}(x_0)$ hold for each $x_0 \in I_0$.*

**Proof.** Suppose that there exists an $x_0$ and a finite $\alpha$ such that $\underline{F}^s(x_0) < \alpha \leq \underline{F}^s_{ap}(x_0)$. Since $F$ is nondecreasing, $\underline{F}^s(x_0) \geq 0$ and so $\alpha > 0$. Choose $\varepsilon > 0$ such that $\underline{F}^s(x_0) < \alpha - 2\varepsilon$. Then there exists a sequence $\{h_n\}$ of positive numbers converging to 0 such that

$$(F(x_0 + h_n) - F(x_0 - h_n))/2h_n < \alpha - 2\varepsilon$$

holds for each $n$.

We shall show for each $n$ that the inequality $(F(x_0+t)-F(x_0-t))/2t < \alpha-\varepsilon$ is valid for all $t$ in $(-h_n, -h_n(1-\varepsilon/(\alpha-\varepsilon)))\cup(h_n(1-\varepsilon/(\alpha-\varepsilon)), h_n)$. Since the relative measure of this set in $[-h_n, h_n]$ is $\varepsilon/(\alpha-\varepsilon)$ and $h_n \to 0$ as $n \to \infty$, it will follow that the upper density at zero of the set $A(\alpha-\varepsilon) = \{t : (F(x_0+t)-F(x_0-t))/2t < \alpha-\varepsilon\}$ is at least $\varepsilon/(\alpha-\varepsilon) > 0$. This will imply that $\underline{F}^s_{ap}(x_0) \leq \alpha - \varepsilon$, a contradiction.

Let $h \in \{h_n\}$. Then, $(F(x_0 + h) - F(x_0 - h))/2h < \alpha - 2\varepsilon$. Put $a = (F(x_0 + h) + F(x_0 - h))/2$ and consider the linear functions $L$ and $L_0$ which satisfy $L(x_0) = L_0(x_0) = a$ and have slopes $\alpha - \varepsilon$ and $\alpha - 2\varepsilon$, respectively. Choose $x_1$ and $x_2$ such that $L(x_1) = F(x_0 + h)$ and $L(x_2) = F(x_0 - h)$. Then obviously

$$\begin{aligned} x_1 &= x_0 + (F(x_0 + h) - a)/(\alpha - \varepsilon) \quad \text{and} \\ x_2 &= x_0 - (a - F(x_0 - h))/(\alpha - \varepsilon). \end{aligned}$$

Since $F$ is nondecreasing we have

$$(*) \qquad \begin{aligned} &F(x) \leq F(x_0 + h) < L(x) \;\text{ for }\; x \in (x_1, x_0 + h) \;\text{ and} \\ &L(x) < F(x_0 - h) \leq F(x) \;\text{ for }\; x \in (x_0 - h, x_2). \end{aligned}$$

To compute the length of intervals $[x_1, x_0 + h]$ and $[x_0 - h, x_2]$ observe that

$$\begin{aligned} x_0 + h - x_1 &= h - (F(x_0 + h) - a)/(\alpha - \varepsilon) \\ &= (h(\alpha - \varepsilon) - F(x_0 + h) + a)/(\alpha - \varepsilon) \\ &= (L(x_0 + h) - F(x_0 + h))/(\alpha - \varepsilon) \\ &> (L(x_0 + h) - L_0(x_0 + h))/(\alpha - \varepsilon) \\ &= h\varepsilon/(\alpha - \varepsilon). \end{aligned}$$

Analogously it can be shown $x_2 - (x_0 - h) > h\varepsilon/(\alpha - \varepsilon)$. Thus

$$\begin{aligned} &[x_0 + h(1 - \varepsilon/(\alpha - \varepsilon)), x_0 + h] \subset [x_1, x_0 + h] \;\text{ and} \\ &[x_0 - h, x_0 - h(1 - \varepsilon/(\alpha - \varepsilon))] \subset [x_0 - h, x_2]. \end{aligned}$$

It follows from $(*)$ that $(F(x_0+t)-F(x_0-t))/2t < (L(x_0+t)-L(x_0-t))/2t = \alpha-\varepsilon$ when $t \in (-h, -h(1 - \varepsilon/(\alpha - \varepsilon))) \cup (h(1 - \varepsilon/(\alpha - \varepsilon)), h)$, as was to be proved.

The argument for the upper approximate symmetric derivative and the upper symmetric derivative is similar. (If $\overline{F}^s_{ap}(x_0) \leq \alpha < \alpha + 2\varepsilon < \overline{F}^s(x_0)$, we deal with $t \in (-h(1 + \varepsilon/(\alpha + \varepsilon)), -h) \cup (h, h(1 + \varepsilon/(\alpha + \varepsilon)))$. Then the upper density at zero of the set $B(\alpha + \varepsilon) = \{t : (F(x_0 + t) - F(x_0 - t))/2t > \alpha + \varepsilon\}$ is at least $\varepsilon/(\alpha + \varepsilon) > 0$ and $\overline{F}^s_{ap}(x_0) \geq \alpha + \varepsilon$, a contradiction.)

**Theorem 1.** *Let $F$ be a nondecreasing function defined on an open interval $I_0$. If $F$ has the approximate symmetric derivative at $x$, then $F$ has the symmetric derivative at $x$ and $F^s(x) = F^s_{ap}(x)$.*

**Proof.** If $F$ has the approximate symmetric derivative at $x$, then $F^s_{ap}(x) = \underline{F}^s_{ap}(x) = \overline{F}^s_{ap}(x)$ and the theorem follows from Lemma.

**Theorem 2.** *Let $F$ be a measurable approximately symmetrically differentiable function on $I_0$ and let $G$ be symmetrically differentiable on $I_0$. If $F^s_{ap}(x) \leq G^s(x)$ everywhere on $I_0$, then there exists a symmetrically differentiable function $f : I_0 \to R$ such that $f(x) = F(x)$ a.e. on $I_0$ and $F^s_{ap}(x) = f^s(x)$ everywhere on $I_0$.*

**Proof.** Since $G$ is symmetrically differentiable on $I_0$, it is symmetrically continuous on $I_0$, i.e. $\lim(G(x_0 + h) - G(x_0 - h)) = 0$ as $h \to 0$ for each $x$ in $I_0$. According to the paper [B], $G$ is continuous almost everywhere on $I_0$ and hence $G$ is measurable. Let $H = G - F$. Then $H$ is measurable and $H^s_{ap}(x) = G^s_{ap}(x) - F^s_{ap}(x) \geq 0$ holds everywhere on $I_0$. It follows from Theorem 2 of [FR] that $H$ is nondecreasing on the set $A$ of all points for which $H$ is approximately continuous. Define the function $h : I_0 \to R$ in the following way: $h(x) = H(x)$ for $x \in A, h(x) = \sup\{H(t) : t < x, t \in A\}$ for $x \in I_0 - A$. The function $h$ is obviously nondecreasing. Since $H$ is measurable, $\lambda(I_0 - A) = 0$ and $\lambda\{x : h(x) \neq H(x)\} = 0$. According to Theorem 1, $h^s(x) = h^s_{ap}(x) = H^s_{ap}(x)$ holds everywhere on $I_0$. Let $f = G - h$. Then $f(x) = F(x)$ a.e. on $I_0$ and $f^s(x) = F^s_{ap}(x)$ everywhere on $I_0$.

**Corollary.** *Let $F$ be a Lebesgue measurable function, approximately symmetrically differentiable on $I_0$. If $F^s_{ap}$ is locally bounded, then there is a Lebesgue measurable, symmetrically differentiable function $f : I_0 \to R$ such that $f(x) = F(x)$ a.e. on $I_0$ and $f^s(x) = F^s_{ap}(x)$ everywhere on $I_0$.*

**Remark.** An approximately symmetrically differentiable function need not be measurable. Assuming the continuum hypothesis, W. Sierpiński has shown that there is a nonmeasurable function $f : R \to R$ with $f^s_{ap}(x) = 0$ for every $x$ in $R$ [S, Corollaire 4].

**Theorem 3.** *The class of all approximate symmetric derivatives of measurable functions is closed with respect to the uniform convergence.*

**Proof.** Let $\{f_n\}_1^\infty$, where $f_n : I_0 \to R$, be a sequence of approximate symmetric derivatives of measurable functions and let $f_n \to f$ uniformly. Then there is an $n_0 > 0$ such that $|f_n - f| < 1$ whenever $n \geq n_0$. Define $g_n = f_n - f_{n_0}$ for $n \geq n_0$. Obviously, $|g_n| < 2$ and $g_n \to f - f_{n_0}$ uniformly. According to the Corollary, $\{g_n\}_{n_0}^\infty$ is a sequence of symmetric derivatives. Since $g_n \to g = f - f_{n_0}$ uniformly, $g$ is a symmetric derivative (see [K]). Hence $f = g + f_{n_0}$ is an approximate symmetric derivative.

**Example.** We show that there is an approximate symmetric derivative of a measurable function that is not simultaneously a symmetric derivative and an approximate derivative.

Let $I_0$ be an open real interval and $x_0 \in I_0$. Let $\{a_n\}$ and $\{b_n\}$ be two sequences in $I_0$ such that inequalities $a_n < b_n < a_{n+1} < x_0$ hold for every $n$, that $a_n \to x_0$ and $d(\bigcup(a_n, b_n), x_0) = 0$. Put $c_n = \frac{1}{2}(a_n + b_n)$ and define a function $F : I_0 \to R$ in the following way: $F(x) = 0$ for $x \notin \bigcup(a_n, b_n)$, $F(c_n) = 1$ for every $n$, $F(x) = (x - a_n)/(c_n - a_n)$ for $x \in (a_n, c_n)$ and $F(x) = -(x - b_n)/(b_n - c_n)$ for $x \in (c_n, b_n)$. Obviously $F$ is Lebesgue measurable and approximately symmetrically differentiable on $I_0$. If $f = F^s_{ap}$, then $f(x) = 0$ for $x \notin \bigcup[a_n, b_n]$ and $x = c_n$, $f(x) = (c_n - a_n)^{-1}$ for $x \in (a_n, c_n)$, $f(x) = -(b_n - c_n)^{-1}$ for $x \in (c_n, b_n)$, $f(x) = (2(c_n - a_n))^{-1}$ for $x = a_n$ and $f(x) = -(2(b_n - c_n))^{-1}$ for $x = b_n$. Since every approximate derivative has the Darboux property (see [Br]) $f$ is not an approximate derivative. We show that $f$ is not a symmetric derivative of measurable function. Suppose that there is a measurable function $G : I_0 \to R$ such that $G^s = f$. Put $H = F - G$. Then $H^s_{ap} = F^s_{ap} - G^s_{ap} = 0$, and according to Theorem 2 of [FR] $H$ is constant on the set $A$ of all points for which $H$ is approximately continuous. Since $H$ is measurable, the set $A$ is of full measure in $I_0$. Without loss of generality we can suppose $H(x) = 0$ everywhere on $A$. Hence $G(x) = F(x)$ everywhere on $A$ and $-\infty = \underline{G}^s(x_0) < \overline{G}^s(x_0) = 0$, a contradiction.

# References


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