Real Analysis Exchange Vol. 17 (1991-92)

Seng-Kee Chua, Department of Mathematics, National University of Singapore, Singapore.

RESTRICTION THEOREMS ON WEIGHTED SOBOLEV SPACES OF MIXED NORM

Abstract

By similar methods as Calderón [1] and Stein [10], we prove some restriction theorems on weighted potential spaces when the weight satisfies Muckenhoupt's A_p condition. We then construct potential spaces of mixed norm and find that mixed norm potential spaces are indeed a natural object in the study of restriction theorems. With the help of mixed norm spaces, we obtain a better version of restriction theorem.

§ 1. Introduction

In this paper, we discuss some restriction theorems on potential spaces. $\mathcal{S}$ will always denote the Schwartz space and C denotes various positive constants which may differ even in the same string of estimate. If γ is a multi-index, $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n) \in \mathbb{Z}_+^n$, we will denote $\sum_{j=1}^n \gamma_j$ by $|\gamma|$ and $D^\gamma = (\frac{\partial}{\partial x_1})^{\gamma_1} \cdots (\frac{\partial}{\partial x_n})^{\gamma_n}$. By $\gamma \geq \gamma^o$, we mean $\gamma_j \geq \gamma_j^o$ for all $1 \leq j \leq n$. Moreover we write $\gamma > \gamma^o$ if $\gamma \geq \gamma^o$ and $\gamma \neq \gamma^o$. Let Q be a cube (or ball). Then CQ will always denote the cube (or ball) with same center as Q and C times its edgelength (or radius). If $1 < p < \infty$, we will let $p' = p/(p-1)$.

1991 Mathematics Subject Classification. Primary 46E35.
Keywords and phrases. A_p weights, Sobolev embedding theorems, Bessel potentials.

We write $w \in A_p$ when w satisfies Muckenhoupt's A_p condition [7]. By $f \in L^p_w(\mathbb{R}^n)$, we mean f is L^p integrable with respect to the measure wdx. Moreover, for $1 \le p < \infty$, $L^p_{w,k}(\mathbb{R}^n)$ denotes the space of functions having weak derivatives of all order α, $|\alpha| \le k$ and satisfying

$$\| f \|_{L^p_{w,k}(\mathbb{R}^n)} = \sum_{0\le|\gamma|\le k} \|D^\gamma f\|_{L^p_w(\mathbb{R}^n)} = \sum_{0\le|\gamma|\le k} \left(\int_{\mathbb{R}^n} | D^\gamma f |^p \, dw\right)^{1/p} < \infty$$

where $D^\gamma f$ is the weak derivative of f of order γ.

Following E.M. Stein [9], we define for each $\alpha > 0$ and $x \in \mathbb{R}^n$,

$$G_{\alpha,n}(x) = (4\pi)^{-\alpha/2} \frac{1}{\Gamma(\alpha/2)} \int_0^\infty e^{-\pi|x|^2/\delta} e^{-\delta/4\pi} \delta^{-(n-\alpha)/2} \frac{d\delta}{\delta}. \tag{1.1}$$

We will write $G_\alpha = G_{\alpha,n}$ when there is no danger of ambiguity. For any $\alpha \ge 0$ and $f \in L^p_w(\mathbb{R}^n)$, $1 < p \le \infty, w \in A_p$, we can define

$$J_\alpha(f) = J_{\alpha,n}(f) = \begin{cases} G_\alpha * f & \text{if } \alpha > 0 \\ f & \text{if } \alpha = 0. \end{cases}$$

Next, following N. Miller [6], we have the following definition of weighted potential spaces.

Definition 1.2 Let $1 < p < \infty$, $w \in A_p$, $\alpha \ge 0$. We write $\mathcal{L}^p_{w,\alpha}(\mathbb{R}^n) = J_\alpha(L^p_w(\mathbb{R}^n))$ with norm $\|f\|_{\mathcal{L}^p_{w,\alpha}} = \|g\|_{L^p_w(\mathbb{R}^n)}$ if $f = J_\alpha g$. [1]

Note that this definition is just the weighted version for potential spaces $\mathcal{L}^p_\alpha$ (see [9, Chapter V, section 3] or [1]). A.P. Calderón [1] proved that

Theorem A *Let $1 < p \le 2$ and $\alpha > (n-m)/p$. If $f \in \mathcal{L}^p_\alpha(\mathbb{R}^n)$ and $\mathbb{R}^m$ is an m-dimensional subspace of $\mathbb{R}^n$, then f coincides almost everywhere on $\mathbb{R}^m$ with a function in $\mathcal{L}^p_\beta(\mathbb{R}^m)$ where $\beta = \alpha - (n-m)/p$. The restriction of f to $\mathbb{R}^m$ induces a continuous mapping of $\mathcal{L}^p_\alpha(\mathbb{R}^n)$ to $\mathcal{L}^p_\beta(\mathbb{R}^m)$.*

Later, E.M. Stein improved Theorem A by showing that the restriction mapping is indeed a continuous mapping of $\mathcal{L}^p_\alpha(\mathbb{R}^n)$ into certain subspace of $\mathcal{L}^p_\beta(\mathbb{R}^m)$ (see [10]). Our purpose is to extend Theorem A to weighted potential spaces when the weight satisfies the Muckenhoupt A_p condition. First, we state some simple facts on weighted potential spaces in section 2. Then we prove some restriction theorems on weighted potential spaces in section 3. Finally, we establish potential spaces of mixed norm and discuss restriction problems on mixed norm potential spaces. With the help of these spaces, we improve results in section 3.

[1] One could also define weighted potential spaces when $p = 1$ and $w \in A_1$; see [8].

§ 2. Weighted potential spaces

In this section, we collect some elementary facts for easy reference. Most of the fact stated in this section can be found in [8], [6] or [3].

First, let us state some simple properties from [9] and [1].

Proposition 2.1 *For each $\alpha > 0$,*

(a) $G_\alpha(x) \in L^1(\mathbb{R}^n)$,

(b) $\hat{G}_\alpha(x) = (1 + 4\pi^2|x|^2)^{-\alpha/2}$ where $\hat{G}_\alpha$ is the Fourier transform of G_α,

(c) G_α is radial and decreases as $|x|$ increases,

(d) $|G_\alpha(x)| \leq C/|x|^{n-\alpha}$ for all $x \in \mathbb{R}^n$, $\alpha < n$,

(e) $G_\alpha(x) = \frac{|x|^{-n+\alpha}}{\gamma(\alpha)} + o(|x|^{-n+\alpha})$ as $|x| \to 0$.

Next, we will state a well-known lemma; see for example, Theorem 9.17 in [12].

Proposition 2.2 *Let $k(x)$ be nonnegative and integrable on $\mathbb{R}^n$ and suppose $k(x)$ depends only on $\mid x \mid$ and decreases as $\mid x \mid$ increases. Then for all non-negative measurable functions f,*

$$f * k(x) \leq C\|k\|_{L^1(\mathbb{R}^n)} f^*(x)$$

with C independent of f and k. Here f^* is the Hardy-Littlewood maximal function of f.

It follows from these two propositions that $J_\alpha(f)$ is well-defined for $f \in L^p_w(\mathbb{R}^n)$ when $1 < p \leq \infty, w \in A_p$, since

$$\|G_\alpha * f\|_{L^p_w} \leq C\|f^*\|_{L^p_w} \leq C\|f\|_{L^p_w} \tag{2.3}$$

by [11, Theorem IX.4.1]. Note that C is independent of α as $\|G_\alpha\|_{L^1} = 1$. Thus, indeed $J_\alpha(f) \in L^p_w(\mathbb{R}^n)$.

Next, let us state a fact from [4] about singular integrals.

Proposition 2.4 *Let K be a real function which satisfies the following three properties:*

(a) $\|\hat{K}\|_\infty \leq C$ ($\hat{K}$ is the Fourier transform of K.)

(b) $|K(x)| \leq C/|x|^n$

(c) $|K(x) - K(x-y)| \leq C|y|/|x|^{n+1}$ *for* $|y| < |x|/2$.
If $w \in A_p$ *and* $1 < p < \infty$, *then*

$$\|K * f\|_{L^p_w(\mathbb{R}^n)} \leq C\|f\|_{L^p_w(\mathbb{R}^n)} \quad \textit{for } f \in L^p_w(\mathbb{R}^n).$$

The following lemma is indeed a consequence of the proof of Lemma 2 in [9, Chapter V] . The reader can also find the details in [3].

Lemma 2.5 *There exists a finite measure* μ *on* $\mathbb{R}^n$ *such that its Fourier transform is*

$$\hat{\mu} = \frac{2\pi|x|}{(1+4\pi^2|x|^2)^{1/2}}.$$

Moreover

$$\|\mu * g\|_{L^p_w} \leq C\|g\|_{L^p_w} \quad \textit{if } 1 < p < \infty \textit{ and } w \in A_p.$$

The next two facts can be found in [8] or [3].

Proposition 2.6 *Let* $1 < p < \infty$, $w \in A_p$ *and* $\alpha \geq 1$. *If* $f \in \mathcal{L}^p_{w,\alpha}(\mathbb{R}^n)$ *then* $f, \frac{\partial f}{\partial x_j} \in \mathcal{L}^p_{w,\alpha-1}(\mathbb{R}^n)$ *for all* j. *Moreover,*

$$\|f\|_{\mathcal{L}^p_{w,\alpha-1}} + \sum_{j=1}^{n} \|\frac{\partial f}{\partial x_j}\|_{\mathcal{L}^p_{w,\alpha-1}} \leq C\|f\|_{\mathcal{L}^p_{w,\alpha}}.$$

Theorem 2.7 *If* k *is a positive integer and* $1 < p < \infty$, $w \in A_p$, *then* $\mathcal{L}^p_{w,k}(\mathbb{R}^n) = L^p_{w,k}(\mathbb{R}^n)$. *Moreover, their norms are equivalent.*

§ 3. Restriction theorems

Let $\psi : \mathbb{R}^{n-1} \to \mathbb{R}$ and $\Omega = \{(x, \psi(x)) : x \in \mathbb{R}^{n-1}\}$. By a weight w, we mean a non-negative locally integrable function on $\mathbb{R}^n$. By abusing notation, we will also write w for the measure induced by w. Sometimes we write dw to denote wdx. Let w be a weight in $\mathbb{R}^{n-1}$. We denote $L^p_w(\Omega)$ to be the collection of all functions f on Ω such that

$$\|f\|_{L^p_w(\Omega)} = (\int_{\mathbb{R}^{n-1}} |f(x, \psi(x))|^p w(x)dx)^{1/p} < \infty.$$

Moreover, let $\psi \in C^k(\mathbb{R}^{n-1})$ and $D^\gamma\psi$ are Lipschitz continuous for all γ, $|\gamma| \leq k-1\}$. Then we said a locally integrable function f on Ω has a weak derivative of order γ, $|\gamma| \leq k$, if there is a locally integrable function (denoted by $D^\gamma f$) such that

$$\int_{\mathbb{R}^{n-1}} f(x,\psi(x))D^\gamma\varphi(x,\psi(x))dx = (-1)^{|\gamma|}\int_{\mathbb{R}^{n-1}} D^\gamma f(x,\psi(x))\varphi(x,\psi(x))dx$$

for all $\varphi \in C_0^k(\Omega)$. Also, we let $L^p_{w,k}(\Omega)$ be the space of functions having weak derivatives of all order γ, $|\gamma| \leq k$ and satisfying

$$\|f\|_{L^p_{w,k}(\Omega)} = \sum_{0\leq|\gamma|\leq k} \|D^\gamma f\|_{L^p_w(\Omega)} < \infty.$$

Now, let us give a sufficient condition for restriction theorem on weighted potential spaces.

Theorem 3.1 *Let $1 < p < \infty$ and let $w \in A_p(\mathbb{R}^n)$, $w_1 \in A_p(\mathbb{R}^{n-1})$. Suppose that $\psi : \mathbb{R}^{n-1} \to \mathbb{R}$ is continuous. Furthermore, assume that there exist $a, a^* \in \mathbb{R}$, $a \leq a^*$ and $C_1, C_2 > 0$ such that for $\delta > 0$ and $x, y \in \mathbb{R}^{n-1}$,*

$$\int_{-\infty}^{\infty} e^{-\pi p'|\psi(x)-\xi|^2/\delta}\left(\frac{w(y,\xi)}{w_1(y)}\right)^{-1/(p-1)} d\xi \leq C_1\delta^{a/2} + C_2\delta^{a^*/2}. \tag{3.2}$$

If $f \in \mathcal{L}^p_{w,\alpha}(\mathbb{R}^n)$, $\alpha > \max(1-\frac{a}{p'},0)$, then f coincides almost everywhere on $\Omega = \{(x,\psi(x)) : x \in \mathbb{R}^{n-1}\}$ with a function in $L^p_{w_1}(\Omega)$. The restriction of f to Ω induces a continuous mapping of $\mathcal{L}^p_{w,\alpha}(\mathbb{R}^n)$ into $L^p_{w_1}(\Omega)$.

Proof First note that by Minkowski's inequality and (3.2), we have

$$\begin{aligned}
&\left(\int_{-\infty}^{\infty} G_\alpha(x-y,\psi(x)-\xi)^{p'}\left(\frac{w(y,\xi)}{w_1(y)}\right)^{-1/(p-1)} d\xi\right)^{1/p'} \\
\leq\ & C\int_0^\infty e^{-\pi|x-y|^2/\delta}e^{-\delta/4\pi}\delta^{(\alpha-n)/2}(\delta^{a/2p'} + \delta^{a^*/2p'})\frac{d\delta}{\delta} \\
=\ & CG_{\beta,n-1}(x-y) + CG_{\beta^*,n-1}(x-y)
\end{aligned} \tag{3.3}$$

where $\beta = \alpha - 1 + a/p' > 0$ and $\beta^* = \alpha - 1 + a^*/p' > 0$. Now, given $f \in \mathcal{L}^p_{w,\alpha}(\mathbb{R}^n)$, let $g \in L^p_w(\mathbb{R}^n)$ such that $f = G_\alpha * g$. Then

$$f(x,\psi(x)) = \int_{\mathbb{R}^{n-1}}\int_{-\infty}^{\infty} G_\alpha(x-y,\psi(x)-\xi)g(y,\xi)d\xi dy,\ x \in \mathbb{R}^{n-1}.$$

Now let $G = G_{\beta,n-1} + G_{\beta^*,n-1}$. Note by Hölder's inequality, (3.3) and (2.3),

$$(\int_{\mathbb{R}^{n-1}}(\int_{\mathbb{R}^{n-1}}\int_{-\infty}^{\infty} G_\alpha(x-y,\psi(x)-\xi)|g(y,\xi)|d\xi dy)^p w_1(x)dx)^{1/p}$$

$$\leq (\int_{\mathbb{R}^{n-1}}\{\int_{\mathbb{R}^{n-1}}(\int_{-\infty}^{\infty}|g(y,\xi)|^p \frac{w(y,\xi)}{w_1(y)}d\xi)^{1/p} \times$$

$$(\int_{-\infty}^{\infty} G_\alpha(x-y,\psi(x)-\xi)^{p'}\left(\frac{w(y,\xi)}{w_1(y)}\right)^{-1/(p-1)} d\xi)^{1/p'}dy\}^p w_1(x)dx)^{1/p}$$

$$\leq C\|\int_{\mathbb{R}^{n-1}}(\int_{-\infty}^{\infty}|g(y,\xi)|^p\frac{w(y,\xi)}{w_1(y)}d\xi)^{1/p}G(\cdot - y)dy\|_{L^p_{w_1}(\mathbb{R}^{n-1})}$$

$$\leq C\|(\int_{-\infty}^{\infty}|g(\cdot,\xi)|^p\frac{w(\cdot,\xi)}{w_1(\cdot)}d\xi)^{1/p}\|_{L^p_{w_1}(\mathbb{R}^{n-1})} = C\|g\|_{L^p_w(\mathbb{R}^n)}.$$

This concludes the proof of the theorem.

Remark 3.4

(a) The conclusion of Theorem 3.1 involves a but not a^*. It shows that the behavior of the integral of (3.2) as $\delta \to 0$ is more important. When $\delta \to \infty$, we need only to assume that the integral does not grow too fast. Indeed, we could even allow the integral to grow exponentially as $\delta \to \infty$; we could replace (3.2) by

$$\int_{-\infty}^{\infty} e^{-\pi p'|\psi(x)-\xi|^2/\delta}\left(\frac{w(y,\xi)}{w_1(y)}\right)^{-1/(p-1)} d\xi \leq C_1\delta^{a/2} + C_2\delta^{a^*/2}e^{C_3\delta}$$

for some $C_3 < p'/4\pi$ with a, a^*, C_1, C_2 as in (3.2).

(b) (3.2) holds for some $a \leq 1$ when $\Omega = \mathbb{R}^{n-1}$ and $w(x,y) = w_1(x)w_2(y)$, $x \in \mathbb{R}^{n-1}, y \in \mathbb{R}, w_2 \in A_p(\mathbb{R})$. For the details, see [3] or the proof of Lemma 4.20.

(c) In the case $w = w_1 = 1$ and $\Omega = \mathbb{R}^{n-1}$, we observe that $a = 1$ in the previous theorem, and hence the restriction of functions in $\mathcal{L}^p_\alpha(\mathbb{R}^n)$ induces a continuous mapping of $\mathcal{L}^p_\alpha(\mathbb{R}^n)$ into $L^p(\mathbb{R}^{n-1})$ when $\alpha > 1/p$. This is indeed just a partial result of Theorem A in the introduction.

Unfortunately, under the assumptions of Theorem 3.1, we do not know whether the restriction of $\mathcal{L}^p_{w,\alpha}(\mathbb{R}^n)$ functions is in $\mathcal{L}^p_{w_1,\alpha+a/p'-1}(\Omega)$ as in the case when $w = w_1 = 1$. However, if $\alpha + a/p' - 1 > k$ for some $k \in \mathbb{N}$, we have the following theorem.

Theorem 3.5 *Under the assumptions of Theorem 3.1, assume further that $\alpha - k > \max(1 - a/p', 0)$ for some $k \in \mathbb{N}$ and $\psi \in C^k(\mathbb{R}^{n-1})$ with $|D^\gamma \psi| \leq M$ for all $|\gamma| \leq k$. Then for $f \in \mathcal{L}^p_{w,\alpha}(\mathbb{R}^n)$, f coincides almost everywhere on Ω with a function in $L^p_{w_1,k}(\Omega)$. Moreover, the restriction of f to Ω induces a continuous mapping from $\mathcal{L}^p_{w,\alpha}(\mathbb{R}^n)$ to $L^p_{w_1,k}(\Omega)$.*

Proof We will only prove the theorem for $k = 1$. Thus, suppose that $\alpha - 1 > \max(1 - a/p', 0)$. Let us define the restriction of $f \in \mathcal{L}^p_{w,\alpha-1}(\mathbb{R}^n)$ by

$$Rf(x) = \int_{\mathbb{R}^{n-1}} \int_{-\infty}^{\infty} G_{\alpha-1}(x - y, \psi(x) - \xi) g(y, \xi) d\xi dy \quad \text{for } x \in \mathbb{R}^{n-1}$$

where $J_{\alpha-1} g = f$.
Claim:

$$\frac{\partial Rf}{\partial x_i} = R\frac{\partial f}{\partial x_i} + \frac{\partial \psi}{\partial x_i} R \frac{\partial f}{\partial x_n} \text{ for all } 1 \leq i < n \text{ and } f \in \mathcal{L}^p_{w,\alpha}(\mathbb{R}^n).$$

First note that the claim is clearly true when $f \in \mathcal{S}$. In general, if $f \in \mathcal{L}^p_{w,\alpha}(\mathbb{R}^n)$, let us choose $\{f_m\} \subset \mathcal{S}$ with $f_m \to f$ in $\mathcal{L}^p_{w,\alpha}(\mathbb{R}^n)$ and hence $\frac{\partial f_m}{\partial x_i} \to \frac{\partial f}{\partial x_i}$ in $\mathcal{L}^p_{w,\alpha-1}(\mathbb{R}^n)$ by Proposition 2.6. Note that since $\frac{\partial f}{\partial x_i} \in \mathcal{L}^p_{w,\alpha-1}(\mathbb{R}^n)$ and $\alpha - 1 > \max(1 - a/p', 0)$, we have $R\frac{\partial f}{\partial x_i} \in L^p_{w_1}(\mathbb{R}^{n-1})$ and $R\frac{\partial f_m}{\partial x_i} \to R\frac{\partial f}{\partial x_i}$ in $L^p_{w_1}(\mathbb{R}^n)$ by Theorem 3.1. Note that $\frac{\partial \psi}{\partial x_i}$ is bounded by hypothesis, it is now easy to see that our claim holds.

Hence,

$$\|Rf\|_{L^p_{w_1,1}} \leq C \sum_{|\gamma| \leq 1} \|D^\gamma f\|_{L^p_{w,\alpha-1}(\mathbb{R}^n)}.$$

This completes the proof of the theorem.
Remark 3.6 For the case $w(x, y) = w_1(x) \times w_2(y), x \in \mathbb{R}^{n-k}, y \in \mathbb{R}^k$ such that $w_1 \in A_p(\mathbb{R}^{n-k})$ and $w_2 \in A_p(\mathbb{R}^k)$, better results are obtained in the next section.

§ 4. Potential spaces of mixed norm

In this section, we define Sobolev spaces of mixed norm. To simplify the notation, we will only define mixed norm with two variables. However, one can easily extend the ideas to mixed norm of l variables, $l \in \mathbb{N}$.

Definition 4.1 *Let* $\tilde{p} = (p_1, p_2)$, $\tilde{k} = (k_1, k_2)$ *and let* $\tilde{n} = (n_1, n_2)$ *with* $n = n_1 + n_2$. *Also let* $w(x) = w_1(x_1) \times w_2(x_2) \quad x = (x_1, x_2), x_1 \in \mathbb{R}^{n_1}, x_2 \in \mathbb{R}^{n_2}$. *Then* $L^{\tilde{p}}_{w,\tilde{k},\tilde{n}}(\mathbb{R}^n)$ *will denote the space of functions having weak derivatives of all order* $\gamma = (\gamma_1, \cdots, \gamma_n)$ *such that*

$$\gamma_1 + \cdots + \gamma_{n_1} \leq k_1 \quad , \gamma_{n_1+1} + \cdots + \gamma_n \leq k_2 \tag{4.2}$$

and satisfying

$$\|f\|_{L^{\tilde{p}}_{w,\tilde{k},\tilde{n}}(\mathbb{R}^n)} = \sum_{\gamma \in \Gamma(\tilde{k},\tilde{n})} \|D^\gamma f\|_{L^{\tilde{p}}_w(\mathbb{R}^n)} < \infty$$

where $\Gamma(\tilde{k}, \tilde{n})$ *is the collection of all* γ *which satisfies (4.2) and*

$$\|D^\gamma f\|_{L^{\tilde{p}}_w(\mathbb{R}^n)} = (\int_{\mathbb{R}^{n_2}} (\int_{\mathbb{R}^{n_1}} |D^\gamma f|^{p_1} dw_1)^{p_2/p_1} dw_2)^{1/p_2}.$$

It follows from the proof of Theorem 3.1 in [2] that $C_0^\infty(\mathbb{R}^n)$ is dense in $L^p_{w,k}(\mathbb{R}^n)$. We will now show that $C_0^\infty(\mathbb{R}^n)$ is also dense in Sobolev spaces of mixed norm.

Proposition 4.3 *Let* $\tilde{p}, \tilde{k}$ *and* $\tilde{n}$ *as above. Assume further that* $1 \leq p_i < \infty, w_i \in A_{p_i}(\mathbb{R}^{n_i})$. *Then* $C_0^\infty(\mathbb{R}^n)$ *is dense in* $L^{\tilde{p}}_{w,\tilde{k},\tilde{n}}(\mathbb{R}^n)$.

Proof Let $f \in L^{\tilde{p}}_{w,\tilde{k},\tilde{n}}(\mathbb{R}^n)$, by repeated applications of Lebesgue's dominated convergence theorem, observe that given any $\varepsilon > 0$, there exists a compact set K such that

$$\|f\|_{L^{\tilde{p}}_{w,\tilde{k},\tilde{n}}(\mathbb{R}^n \setminus K)} \leq \varepsilon.$$

Next, let $\psi \in C_0^\infty$ such that $\chi_K \leq \psi \leq \chi_{K'}$ where $K' = \{x \in \mathbb{R}^n : dist(x, K) \leq 1\}$. Moreover, let us assume $|D^\gamma \psi| \leq M$ for all $\gamma \in \Gamma(\tilde{k}, \tilde{n})$. Let $p_0 = \max\{p_1, p_2\}$ and $\tilde{p}_0 = (p_0, p_0)$. Now, let us choose $\varphi \in C_0^\infty$ such that φ is radial, nonnegative, decreases as $|x|$ increases and $\int_{\mathbb{R}^n} \varphi = 1$. Then by standard arguments (see for example [2]), we have

$$\varphi_t * f \to f \text{ in } L^{\tilde{p}_0}_{w,\tilde{k},\tilde{n}}(K') \text{ as } t \to 0$$

where $\varphi_t = t^{-n}\varphi(x/t)$. Hence there exists $t > 0$ such that

$$\|\varphi_t * f - f\|_{L^{\tilde{p}}_{w,\tilde{k},\tilde{n}}(K')} \leq C\|\varphi_t * f - f\|_{L^{\tilde{p}_0}_{w,\tilde{k},\tilde{n}}(K')} \leq \varepsilon.$$

Hence for such t,

$$\begin{aligned}
&\|(\varphi_t * f)\psi - f\|_{L^{\tilde p}_{w,\tilde k,\tilde n}(\mathbb{R}^n)}\\
\le\ & \|(\varphi_t * f)\psi - f\psi\|_{L^{\tilde p}_{w,\tilde k,\tilde n}(\mathbb{R}^n)} + \|f\psi - f\|_{L^{\tilde p}_{w,\tilde k,\tilde n}(\mathbb{R}^n)}\\
\le\ & CM\|\varphi_t * f - f\|_{L^{\tilde p}_{w,\tilde k,\tilde n}(K')} + CM\|f\|_{L^{\tilde p}_{w,\tilde k,\tilde n}(\mathbb{R}^n\setminus K)} \le C\varepsilon.
\end{aligned}$$

This concludes the proof of this proposition.

Let $\tilde\alpha = (\alpha_1, \alpha_2)$ and $\tilde n = (n_1, n_2)$. We define

$$G_{\tilde\alpha,\tilde n}(x) = G_{\alpha_1,n_1}(x_1)G_{\alpha_2,n_2}(x_2) \quad \text{where } x = (x_1, x_2), x_1 \in \mathbb{R}^{n_1}, x_2 \in \mathbb{R}^{n_2}.$$

(See (1.1) for the definition of G_{α_i,n_i}.)

For simplicity, from now on, in most cases, we will assume $n = 2$, $n_1 = n_2 = 1$ and $\tilde\alpha = (\alpha_1, \alpha_2)$. We now write $G_{\tilde\alpha}$ instead of $G_{\tilde\alpha,\tilde n}$ and define

$$J_{\tilde\alpha}f(x) = J_{\tilde\alpha,\tilde n}f(x) = \begin{cases} G_{\tilde\alpha} * f(x) & \text{if } \alpha_1, \alpha_2 > 0,\\ f(x) & \text{if } \alpha_1 = \alpha_2 = 0,\\ \int_{\mathbb{R}} G_{\alpha_1}(x_1 - y_1)f(y_1, x_2)dy_1 & \text{if } \alpha_2 = 0, \alpha_1 > 0,\\ \int_{\mathbb{R}} G_{\alpha_2}(x_2 - y_2)f(x_1, y_2)dy_2 & \text{if } \alpha_1 = 0, \alpha_2 > 0. \end{cases}$$

Next, let $w = w_1 \times w_2, 1 < p_1, p_2 < \infty$ and $w_i \in A_{p_i}(\mathbb{R})$ for $i = 1, 2$. Recall that by $f \in L^{\tilde p}_w(\mathbb{R}^2), \tilde p = (p_1, p_2)$, we mean f is measurable on $\mathbb{R}^2$ and

$$\|f\|_{L^{\tilde p}_w} = (\int(\int |f(x_1, x_2)|^{p_1} w_1(x_1)dx_1)^{p_2/p_1} w_2(x_2)dx_2)^{1/p_2} < \infty.$$

The following lemma is an easy consequence of (2.3) and Minkowski's inequality.

Lemma 4.4 *Let $\alpha_1, \alpha_2 \ge 0$ and let p_i, w_i be as above. Then*

$$\|J_{\tilde\alpha}f\|_{L^{\tilde p}_w} \le C\|f\|_{L^{\tilde p}_w} \quad \forall f \in L^{\tilde p}_w(\mathbb{R}^2)$$

where C is a constant independent of $\tilde\alpha$ and f.

Lemma 4.5 *If $f \in L^{\tilde p}_w(\mathbb{R}^2), 1 < p_i < \infty, w_i \in A_{p_i}$ and $\varphi \in \mathcal{S}(\mathbb{R}^2)$ then $f\varphi \in L^1(\mathbb{R}^2)$.*

Proof By Hölders inequality, it suffices to show that $\mathcal{S} \subset L^{\bar{p}}_u$ when $u = u_1 \times u_2$, with u_1, u_2 doubling weights in $\mathbb{R}$. First, note that there exists $C_0 > 0$ such that $u_i(2Q) < C_0 u_i(Q)$ for all intervals Q in $\mathbb{R}$, $i = 1,2$. We now choose m such that $C_0 < 2^{mp_i}$ for $i = 1,2$. Observe that if $\varphi \in \mathcal{S}$, then there exists $A > 0$ such that

$$|\varphi(x_1,x_2)| \le \min(A, \frac{A}{|x_1|^m}, \frac{A}{|x_1|^m|x_2|^m}, \frac{A}{|x_2|^m}).$$

Hence,

$$\begin{aligned}
&\|\varphi\|^{p_2}_{L^{\bar{p}}_u} \\
\le\ & C\int_{|x_2|<1}(\int_{|x_1|<1} A^{p_1}du_1)^{p_2/p_1}du_2 + C\int_{|x_2|\ge 1}(\int_{|x_1|<1}\frac{A^{p_1}}{|x_2|^{mp_1}}du_1)^{p_2/p_1}du_2 \\
+\ & C\int_{|x_2|\ge 1}(\int_{|x_1|\ge 1}\frac{A^{p_1}}{(|x_1||x_2|)^{mp_1}}du_1)^{p_2/p_1}du_2 \\
+\ & C\int_{|x_2|<1}(\int_{|x_1|\ge 1}\frac{A^{p_1}}{|x_1|^{mp_1}}du_1)^{p_2/p_1}du_2 \\
<\ & \infty.
\end{aligned}$$

We will only show that

$$I = \int_{|x_1|\ge 1}\frac{1}{|x_1|^{mp_1}}du_1 < \infty.$$

To see this, let $Q_1 = \{x \in \mathbb{R} : |x| \le 1\}$. Then

$$\begin{aligned}
I &\le \sum_{k=0}^{\infty}\frac{1}{2^{mp_1k}}\int_{2^{k+1}Q_1\setminus 2^kQ_1} du_1 \le \sum \frac{C_0^{k+1}}{2^{mp_1k}}u_1(Q_1) \\
&\le C_0\sum\left(\frac{C_0}{2^{mp_1}}\right)^k u(Q_1) < \infty.
\end{aligned}$$

Next, we also have

Lemma 4.6 *If $g_1, g_2 \in L^{\bar{p}}_w(\mathbb{R}^2), 1 < p_i < \infty, w_i \in A_{p_i}$ and $J_{\tilde{\alpha}}g_1 = J_{\tilde{\alpha}}g_2$ for some $\tilde{\alpha}$, then $g_1 = g_2$ a.e..*

Proof First let $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R})$, observe that

$$\begin{aligned}
&\int\int(J_{\tilde{\alpha}}g_1(x_1,x_2))\varphi_1(x_1)\varphi_2(x_2)dx_1dx_2 \\
=\ &\int\int g_1(x_1,x_2)J_{\alpha_1}\varphi_1(x_1)J_{\alpha_2}\varphi_2(x_2)dx_1dx_2
\end{aligned}$$

and

$$\int\int (J_{\bar{\alpha}} g_2(x_1, x_2))\varphi_1(x_1)\varphi_2(x_2)dx_1 dx_2$$
$$= \int\int g_2(x_1, x_2) J_{\alpha_1}\varphi_1(x_1) J_{\alpha_2}\varphi_2(x_2)dx_1 dx_2.$$

Hence for almost every x_2,

$$\int g_1(x_1, x_2) J_{\alpha_1}\varphi_1(x_1)dx_1 = \int g_2(x_1, x_2) J_{\alpha_1}\varphi_1(x_1)dx_1$$

since $J_{\alpha_2}\mathcal{S}(\mathbb{R}) = \mathcal{S}(\mathbb{R})$ (see for example [9]). Thus $g_1 = g_2$ a.e..

We can now define potential spaces of mixed norm.

Definition 4.7 *Let* $1 < p_i < \infty, w_i \in A_{p_i}, \alpha_i \geq 0$ *for* $i = 1,2$. *We write* $\mathcal{L}^{\bar{p}}_{w,\bar{\alpha},\bar{n}}(\mathbb{R}^2) = \mathcal{L}^{\bar{p}}_{w,\bar{\alpha}}(\mathbb{R}^2) = J_{\bar{\alpha}}(L^{\bar{p}}_w(\mathbb{R}^2))$ *with norm* $\|f\|_{\mathcal{L}^{\bar{p}}_{w,\bar{\alpha}}} = \|g\|_{L^{\bar{p}}_w}$ *if* $f = J_{\bar{\alpha}} g$.

Let us state some simple properties regarding mixed norm potential spaces.

Proposition 4.8 *If* $1 < p_i < \infty$ *and* $w_i \in A_{p_i}$ *for* $i = 1,2$, *then*
(a) $\mathcal{L}^{\bar{p}}_{w,\bar{\beta}} \subset \mathcal{L}^{\bar{p}}_{w,\bar{\alpha}}$ *and* $\|f\|_{\mathcal{L}^{\bar{p}}_{w,\bar{\alpha}}} \leq \|f\|_{\mathcal{L}^{\bar{p}}_{w,\bar{\beta}}}$ *if* $\alpha_i \leq \beta_i$ *for all* i,
(b) $J_{\bar{\beta}}$ *is an isomorphism of* $\mathcal{L}^{\bar{p}}_{w,\bar{\alpha}}$ *to* $\mathcal{L}^{\bar{p}}_{w,\bar{\alpha}+\bar{\beta}}$.

Next, similar to Theorem 2.7, we have

Theorem 4.9 *If* $k_i \in \mathbb{Z}_+, 1 < p_i < \infty, w_i \in A_{p_i}$ *for* $i = 1,2$, *then* $\mathcal{L}^{\bar{p}}_{w,\bar{k}} = L^{\bar{p}}_{w,\bar{k}}$. *Moreover, their norms are equivalent.*

First, similar to Proposition 2.6, we have the following lemma. Note that the proof can be done in a similar way as [10, Lemma V.3], using the fact that the Riesz transform is bounded on L^p_w if $w \in A_p$ and using Lemma 2.5, Minkowski's inequality and Propositions 2.4 and 4.3.

Lemma 4.10 *Let* $1 < p_i < \infty, w \in A_{p_i}$. *If* $f \in \mathcal{L}^{\bar{p}}_{w,\bar{\alpha}}(\mathbb{R}^2)$, *then* $\frac{\partial f}{\partial x_1} \in \mathcal{L}^{\bar{p}}_{w,(\alpha_1-1,\alpha_2)}$ *when* $\alpha_1 \geq 1$ *and* $\frac{\partial f}{\partial x_2} \in \mathcal{L}^{\bar{p}}_{w,(\alpha_1,\alpha_2-1)}$ *when* $\alpha_2 \geq 1$. *Moreover,*

$$\|\frac{\partial f}{\partial x_1}\|_{\mathcal{L}^{\bar{p}}_{w,(\alpha_1-1,\alpha_2)}} + \|\frac{\partial f}{\partial x_2}\|_{\mathcal{L}^{\bar{p}}_{w,(\alpha_1,\alpha_2-1)}} \leq C\|f\|_{\mathcal{L}^{\bar{p}}_{w,\bar{\alpha}}}.$$

Proof of Theorem 4.9 Applying Lemma 4.10, we note by induction that it suffices to show that if $f \in L^{\tilde{p}}_{w,\tilde{k}}$ then $f \in \mathcal{L}^{\tilde{p}}_{w,\tilde{k}}$ and $\|f\|_{\mathcal{L}^{\tilde{p}}_{w,\tilde{k}}} \leq C\|f\|_{L^{\tilde{p}}_{w,\tilde{k}}}$. Let us choose $s_1, s_2 \in \mathbb{Z}_+$ such that $k_i \leq 2s_i$ and let $r_i = 2s_i - k_i$ for $i = 1, 2$. If $\tilde{r} = (r_1, r_2)$, then

$$\|D^\beta J_{\tilde{r}}(D^\gamma f)\|_{L^{\tilde{p}}_w} \leq C\|D^\gamma f\|_{L^{\tilde{p}}_w} \leq C\|f\|_{L^{\tilde{p}}_{w,\tilde{k}}} \quad \text{for } \beta \leq \tilde{r} \text{ and } \gamma \leq \tilde{k}. \tag{4.11}$$

Next if $\alpha_i \leq 2s_i$, let us choose $\beta \leq \tilde{r}, \gamma \leq \tilde{k}$ such that $\beta + \gamma = \alpha$. We claim that $D^\alpha J_{\tilde{r}}(f) = D^\beta J_{\tilde{r}}(D^\gamma f)$ (as distributions) for all $f \in L^{\tilde{p}}_{w,\tilde{k}}$. The claim is clearly true when $f \in \mathcal{S}$. Since $\mathcal{S}$ is dense in $L^{\tilde{p}}_{w,\tilde{k}}$, we can prove the claim by taking the limit and using (4.11).

Next

$$\begin{aligned} &\|(1 - \left(\frac{\partial}{\partial x_1}\right)^2)^{s_1}(1 - \left(\frac{\partial}{\partial x_2}\right)^2)^{s_2} J_{\tilde{r}} f\|_{L^{\tilde{p}}_w} \\ \leq &\sum_{\alpha_1 \leq 2s_1, \alpha_2 \leq 2s_2} C_\alpha \|D^\alpha J_{\tilde{r}} f\|_{L^{\tilde{p}}_w} \\ \leq &\sum_{\beta \leq \tilde{r}, \gamma \leq \tilde{k}} \|D^\beta J_{\tilde{r}}(D^\gamma f)\|_{L^{\tilde{p}}_w} \leq C\|f\|_{L^{\tilde{p}}_{w,\tilde{k}}}. \end{aligned} \tag{4.12}$$

Finally, let us observe that if $f \in L^p_{w,k}(\mathbb{R}^n)$, then $J_{\tilde{k}}(1 - \left(\frac{\partial}{\partial x_1}\right)^2)^{s_1}(1 - \left(\frac{\partial}{\partial x_2}\right)^2)^{s_2} J_{\tilde{r}} f = f$ a.e.. This is clearly true if $f \in C_0^\infty$. In the general case, let us choose a sequence $\{f_m\}$ in C_0^∞ such that $f_m \to f$ in $L^{\tilde{p}}_{w,\tilde{k}}$ and $f_m \to f$ a.e.. Then by (4.12), the conclusion follows. This concludes the proof of Theorem 4.9.

It follows from Theorem 4.9 that $\mathcal{L}^p_{w,k_1+k_2} \subset \mathcal{L}^{\tilde{p}}_{w,(k_1,k_2)}$ when $\tilde{p} = (p, p)$ and $k_1, k_2 \in \mathbb{Z}_+$. Moreover, we also have the following theorem.

Theorem 4.13 *Let $\tilde{p} = (p, p), w = w_1 \times w_2$ and $1 < p < \infty, w_i \in A_p$ for $i = 1, 2$. Then $\mathcal{L}^p_{w,\alpha_1+\alpha_2} \subset \mathcal{L}^{\tilde{p}}_{w,(\alpha_1-\varepsilon_1,\alpha_2-\varepsilon_2)}$ for any $\varepsilon_1, \varepsilon_2 > 0$.*

Proof Let $f = J_{\alpha_1+\alpha_2} g$ with $g \in \mathcal{S}$. Then

$$\begin{aligned} \hat{f}(x_1, x_2) &= (1 + 4\pi^2(|x_1|^2 + |x_2|^2))^{-(\alpha_1+\alpha_2)/2}\hat{g}(x_1, x_2) \\ &= (1 + 4\pi^2|x_1|^2)^{-(\alpha_1-\varepsilon_1)/2}(1 + 4\pi^2|x_2|^2)^{-(\alpha_2-\varepsilon_2)/2} \times \end{aligned}$$

$$(1+4\pi^2|x_1|^2)^{-\varepsilon_1/2}(1+\frac{4\pi^2|x_2|^2}{1+4\pi^2|x_1|^2})^{-\alpha_1/2}\times$$
$$(1+4\pi^2|x_2|^2)^{-\varepsilon_2/2}(1+\frac{4\pi^2|x_1|^2}{1+4\pi^2|x_2|^2})^{-\alpha_2/2}\hat{g}(x_1,x_2).$$

We now let

$$\hat{K}_1(x_1,x_2)=(1+4\pi^2|x_1|^2)^{-\varepsilon_1/2}(1+\frac{4\pi^2|x_2|^2}{1+4\pi^2|x_1|^2})^{-\alpha_1/2},$$

and

$$\hat{K}_2(x_1,x_2)=(1+4\pi^2|x_2|^2)^{-\varepsilon_2/2}(1+\frac{4\pi^2|x_1|^2}{1+4\pi^2|x_2|^2})^{-\alpha_2/2}.$$

Then as in [1, pp 41-43], we have (note that we may assume $\alpha_1,\alpha_2<2$)

$$K_1(x_1,x_2)\le Ce^{-|x_2|/2\pi}|x_2|^{\varepsilon_1}\min(|x_2|^{-2},|x_1|^{-2})$$

and a similar estimate for K_2. Thus if $\tilde{\alpha}=(\alpha_1-\varepsilon_1,\alpha_2-\varepsilon_2)$,

$$f=J_{\tilde{\alpha}}(K_1*(K_2*g))\quad\forall\, g\in\mathcal{S}.$$

Let us note that

$$\|J_{\tilde{\alpha}}(K_1*(K_2*g))\|_{L^{\bar{p}}_w}\le C\|K_1*K_2*g\|_{L^{\bar{p}}_w}\le C\|K_2*g\|_{L^{\bar{p}}_w}\le C\|g\|_{L^{\bar{p}}_w}. \quad (4.14)$$

To see (4.14) holds, let us just demonstrate why $\|K_1*g\|_{L^{\bar{p}}_w}\le C\|g\|_{L^{\bar{p}}_w}$. The rest of the proof of (4.14) are about the same. First observe that for each y_2, $H_1(y_1,y_2)=e^{-|y_2|/2\pi}\times|y_2|^{\varepsilon_1}\min(|y_1|^{-2},|y_2|^{-2})$ is radial, nonnegative and decreases as $|y_1|$ increases, we can apply Proposition 2.2 and Theorem IX.4.1 in [11] to see that

$$\begin{aligned}
&\|\int K_1(y_1,y_2)g(\cdot-y_1,x_2-y_2)dy_1\|_{L^{p_1}_{w_1}}\\
\le\;& C\|H_1(\cdot,y_2)\|_{L^1}\|g(\cdot,x_2-y_2)\|_{L^{p_1}_{w_1}}\\
\le\;& Ce^{-|y_2|/2\pi}|y_2|^{\varepsilon_1-1}\|g(\cdot,x_2-y_2)\|_{L^{p_1}_{w_1}}.
\end{aligned}$$

We now apply Minkowski's inequality and proceed as before,

$$\begin{aligned}
&\|K_1*g\|_{L^{\bar{p}}_w}\\
\le\;& (\int(\int\|\int K_1(y_1,y_2)g(\cdot-y_1,x_2-y_2)dy_1\|_{L^{p_1}_{w_1}}dy_2)^{p_2}w_2(x_2)dx_2)^{1/p_2}\\
\le\;& C\|g\|_{L^{\bar{p}}_w}
\end{aligned}$$

since $\int e^{-|y_2|/2\pi}|y_2|^{\varepsilon_1-1}dy_2 < \infty$.

But $\mathcal{S}$ is dense in L^p_w and $\|J_{\tilde{\alpha}}(K_1 * K_2 * g)\|_{L^p_w} \le C\|g\|_{L^p_w}$, it follows that $f = J_{\tilde{\alpha}}(K_1 * K_2 * g)$ for all $g \in L^p_w$ such that $f = J_{\alpha_1+\alpha_2}g$. It is now easy to see that if $f \in \mathcal{L}^p_{w,\alpha_1+\alpha_2}$ then $f \in \mathcal{L}^{\tilde{p}}_{w,(\alpha_1-\varepsilon_1,\alpha_2-\varepsilon_2)}$ for any $\varepsilon_1, \varepsilon_2 > 0$.

Remark 4.15

(a) Unfortunately, we are unable to show that $\mathcal{L}^p_{w,\alpha_1+\alpha_2} \subset \mathcal{L}^{\tilde{p}}_{w,(\alpha_1,\alpha_2)}$ although it seems very likely that it is true.

(b) Although we have only studied the case $n = 2$ and $\tilde{n} = \tilde{k} = (1,1)$, it is easy to see that all results proved above holds for arbitrary $\tilde{n}$ and $\tilde{k}$.

Next, we also have Sobolev embedding theorem for mixed norm spaces.

Theorem 4.16 *Let $\tilde{\alpha} \ge \tilde{\beta} > 0$, $1 < p_i \le q_i < \infty$ and $w_i \in A_{p_i}, v_i \in A_{q_i}$ for all i. Suppose further that for each i, there exists $0 \le \delta_i \le \alpha_i - \beta_i$ such that*

$$|Q|^{\frac{\delta_i}{n_i}-1} v_i(Q)^{1/q_i} \left(\int_Q w_i^{\frac{-1}{p_i-1}}\right)^{1/p_i'} \le C_i$$

for all cubes Q in $\mathbb{R}^{n_i}$. Then $\mathcal{L}^{\tilde{p}}_{w,\tilde{\alpha},\tilde{n}} \subset \mathcal{L}^{\tilde{q}}_{v,\tilde{\beta},\tilde{n}}$ and

$$\|f\|_{\mathcal{L}^{\tilde{q}}_{v,\tilde{\beta},\tilde{n}}} \le C\|f\|_{\mathcal{L}^{\tilde{p}}_{w,\tilde{\alpha},\tilde{n}}}$$

where C depends only on C_i, $i = 1, \cdots, l$.

Proof Let $\tilde{\delta} = (\delta_1, \cdots, \delta_l)$. It suffices to show that

$$\|J_{\tilde{\delta},\tilde{n}} f\|_{\mathcal{L}^{\tilde{q}}_v} \le C\|f\|_{\mathcal{L}^{\tilde{p}}_w}.$$

However, this follows from the fact

$$|J_{\tilde{\delta},\tilde{n}} f(x)| \le C \int \frac{|f(y)|}{|(x-y)^{\tilde{n}-\tilde{\delta}}|} dy,$$

Theorem 1 in [13] and the Minkowski inequality.

We will now study restriction theorems on mixed norm potential spaces.

Theorem 4.17 *Let* $1 < p_i < \infty$ *and* $w_i \in A_{p_i}(\mathbb{R}^{n_i})$ *for* $i = 1, \cdots, l$, $\tilde{n} = (n_1, \cdots, n_l)$. *Assume further that* $G_{\alpha_j, n_j}(b - \cdot) \in L^{p'_j}_{w'_j}(\mathbb{R}^{n_j})$ *for some* $b \in \mathbb{R}^{n_j}$ *where* $w'_j = w_j^{-\frac{1}{p_j - 1}}$. *If* $f \in \mathcal{L}^{\tilde{p}}_{w, \tilde{\alpha}, \tilde{n}}$, *then* f *coincides almost everywhere on* $\mathbb{R}^{n-n_j} = \mathbb{R}^{a_j} \times \{b\} \times \mathbb{R}^{n-n_j-a_j}$, $a_j = \sum_{i<j} n_i$, *with a function in* $\mathcal{L}^{\hat{\tilde{p}}}_{\hat{w}, \hat{\tilde{\alpha}}, \hat{\tilde{n}}}(\mathbb{R}^{n-n_j})$ *where* $\hat{\tilde{p}} = (p_1, \cdots, p_{j-1}, p_{j+1}, \cdots, p_l)$ *and* $\hat{w}, \hat{\tilde{\alpha}}, \hat{\tilde{n}}$ *are defined similarly. Moreover, the restriction of* f *to* $\mathbb{R}^{n-n_j}$ *induces a continuous mapping of* $\mathcal{L}^{\tilde{p}}_{w, \tilde{\alpha}, \tilde{n}}(\mathbb{R}^n)$ *into* $\mathcal{L}^{\hat{\tilde{p}}}_{\hat{w}, \hat{\tilde{\alpha}}, \hat{\tilde{n}}}(\mathbb{R}^{n-n_j})$.

Proof To simplify the notations, we will only prove a typical case: $l = 2, j = 2$, then $\tilde{n} = (n_1, n_2), \tilde{\alpha} = (\alpha_1, \alpha_2), \tilde{p} = (p_1, p_2), w = w_1 \times w_2$, and $\hat{\tilde{p}} = p_1, \hat{w} = w_1, \hat{\tilde{\alpha}} = \alpha_1$. As before, let $g \in L^{\tilde{p}}_w(\mathbb{R}^n)$ such that $f = J_{\tilde{\alpha}} g$, observe that

$$f(x, b) = \int_{\mathbb{R}^{n_2}} \int_{\mathbb{R}^{n_1}} G_{\alpha_1, n_1}(x - y) G_{\alpha_2, n_2}(b - \xi) g(y, \xi) dy d\xi.$$

Let us define the restriction Rf of f as above, i.e., $Rf(x) = f(x, b)$. Then

$$\begin{aligned}
& \|Rf\|_{L^{p_1}_{w_1}} \\
= \; & \Big(\int_{\mathbb{R}^{n_1}} \Big| \int_{\mathbb{R}^{n_2}} \int_{\mathbb{R}^{n_1}} G_{\alpha_1, n_1}(x - y) G_{\alpha_2, n_2}(b - \xi) g(y, \xi) dy d\xi \Big|^{p_1} w_1(x) dx\Big)^{1/p_1} \\
\leq \; & \int_{\mathbb{R}^{n_2}} \Big(\int_{\mathbb{R}^{n_1}} \Big| \int_{\mathbb{R}^{n_1}} G_{\alpha_1, n_1}(x - y) G_{\alpha_2, n_2}(b - \xi) g(y, \xi) dy \Big|^{p_1} w_1(x) dx\Big)^{1/p_1} d\xi \\
& \text{by Minkowski's inequality} \\
\leq \; & C \int_{\mathbb{R}^{n_2}} G_{\alpha_2, n_2}(b - \xi) \Big(\int_{\mathbb{R}^{n_1}} |g(x, \xi)|^{p_1} w_1(x) dx\Big)^{1/p_1} d\xi \\
\leq \; & C \|G_{\alpha_2, n_2}(b - \cdot)\|_{L^{p'_2}_{w'_2}(\mathbb{R}^{n_2})} \|g\|_{L^{\tilde{p}}_w(\mathbb{R}^n)}
\end{aligned} \tag{4.18}$$

by Hölder's inequality. Next, observe that

$$R J_{\tilde{\alpha}, \tilde{n}} g = J_{\alpha_1, n_1} R J_{(0, \alpha_2), \tilde{n}} g.$$

It is now easy to see that our result holds.

Remark 4.19 The assumption on G is optimal by the following reversed Hölder's inequality:

$$\sup \int fg = \|f\|_{L^{p'}_{w'}}$$

where the supremum is taken over $g \in L^p_w$ with $\|g\|_{L^p_w} \leq 1$. (Just take $g = \dfrac{sgn f |f|^{p'/p} w'}{\|f\|^{p'/p}_{L^{p'}_{w'}}}$ where sgn is the sign function.)

Thus if $G_{\alpha_j,n_j}(b - \cdot) \notin L^{p'_j}_{w'_j}(\mathbb{R}^{n_j})$, there exists a sequence $\{g_m\} \subset L^{\tilde{p}}_w(\mathbb{R}^n)$, $\|g_m\|_{L^{\tilde{p}}_w} = 1$, such that $\|RJ_{\tilde{\alpha}} g_m\|_{L^{\hat{p}}_{\hat{w},\hat{n}}(\mathbb{R}^{n-n_j})} \to \infty$ as $m \to \infty$.

Next let us give a sufficient condition for $G_{\alpha,k}(b - \cdot) \in L^p_v(\mathbb{R}^k)$.

Lemma 4.20 *Let $v \in A_p(\mathbb{R}^k)$ and $v \in RD_a(\mathbb{R}^k)$, i.e., there exists $C > 0$ such that*

$$v(\delta B) \leq C\delta^{ak} v(B) \ \ \textit{for all balls } B \textit{ in } \mathbb{R}^k \textit{ and all } 0 < \delta < 1.$$

Then $G_{\alpha,k}(b - \cdot) \in L^p_v(\mathbb{R}^k)$ for $b \in \mathbb{R}^k$ when $\alpha > k(1 - \frac{a}{p})$.

Proof First note that there exist $C, A > 0$ such that

$$e^{-\pi p|\xi|^2/\delta} \leq C\left(\frac{\delta}{\pi|\xi|^2}\right)^{kp/2} \quad \forall\ |\xi| \geq A\sqrt{\delta}.$$

Also, since $v \in A_p$, we know $v \in A_r$ for some $1 < r < p$ (see for example, Theorem 5.5 and Proposition 4.5 in Chapter IX of [11]) and hence $v(MB) \leq CM^{rk}v(B)$ $\forall M \geq 1$ and all balls B where C is some constant ≥ 1. Thus

$$\begin{aligned}
\int_{\mathbb{R}^k} e^{-\pi p|\xi-b|^2/\delta} v(\xi)d\xi &\leq C\int_{|\xi-b|\geq A\sqrt{\delta}} \frac{\delta^{kp/2}v(\xi)}{|\xi-b|^{kp}}d\xi + \int_{|\xi-b|\leq A\sqrt{\delta}} v(\xi)d\xi \\
&\leq C\sum_{j=1}^{\infty}\int_{2^jB_1\setminus 2^{j-1}B_1}\left(\frac{\delta}{|\xi-b|^2}\right)^{kp/2} v(\xi)d\xi + v(B_1) \\
&\quad \text{where } B_1 = \{\xi \in \mathbb{R}^k : |\xi - b| \leq A\sqrt{\delta}\}\ . \\
&\leq C\sum_{j=1}^{\infty}\frac{1}{2^{(j-1)kp}}v(2^jB_1) + v(B_1) \\
&\leq C\sum_{j=1}^{\infty}\frac{1}{2^{jkp}}2^{jkr}v(B_1) + v(B_1) \\
&\leq Cv(B_1) \leq C(\delta^{ka/2} + \delta^{kp/2})
\end{aligned}$$

by considering the case $\delta < 1$ and the case $\delta \geq 1$. Finally, note that by Minkowski's inequality,

$$\begin{aligned}\|G_{\alpha,k}(b-\cdot)\|_{L^p_v(\mathbb{R}^k)} &\leq \int_0^\infty e^{-\delta/4\pi}\delta^{(\alpha-k)/2}\Big(\int_{\mathbb{R}^k} e^{-\pi p|b-\xi|^2/\delta}v(\xi)d\xi\Big)^{1/p}\frac{d\delta}{\delta} \\ &\leq C\int_0^\infty e^{-\delta/4\pi}\delta^{(\alpha-k)/2}(\delta^{ak/2p}+\delta^{k/2})\frac{d\delta}{\delta} < \infty\end{aligned}$$

if $\alpha - k(1-a/p)$ and $\alpha > 0$. However, a always ≤ 1. This completes the proof of this lemma.

Notation Let $A(\mathbb{R}^n)$ and $B(\mathbb{R}^m)$ be two Sobolev spaces. By $A(\mathbb{R}^n) \xrightarrow{R} B(\mathbb{R}^m)$, we mean the restriction of functions in $A(\mathbb{R}^n)$ on $\mathbb{R}^m$ induces a continuous mapping of $A(\mathbb{R}^n)$ to $B(\mathbb{R}^m)$.

Corollary 4.21 *Let $w, \tilde{p}, \tilde{\alpha}, \tilde{n}, \hat{w}, \hat{\tilde{p}}, \hat{\tilde{\alpha}}, \hat{\tilde{n}}$ be as in Theorem 4.17 and $w_j^{-1/(p_j-1)} \in RD_a(\mathbb{R}^k)$, $\alpha_j > k(1-\frac{a}{p_j'})$, $k = n_j$. Then $\mathcal{L}^{\tilde{p}}_{w,\tilde{\alpha},\tilde{n}}(\mathbb{R}^n) \xrightarrow{R} \mathcal{L}^{\hat{\tilde{p}}}_{\hat{w},\hat{\tilde{\alpha}},\hat{\tilde{n}}}(\mathbb{R}^{n-k})$ where $\mathbb{R}^{n-k} = \mathbb{R}^{a_j} \times \{b\} \times \mathbb{R}^{n-k-a_j}$, $a_j = \sum_{i<j} n_i$ for any $b \in \mathbb{R}^k$.*

Remark 4.22(a) Since $w_j^{-1/(p_j-1)} \in A_{p_j'}(\mathbb{R}^k)$, let us note that there exists $0 < a \leq 1$ such that $w_j^{-1/(p_j-1)} \in RD_a(\mathbb{R}^k)$.

(b) The assumption $\alpha_j > k(1-\frac{a}{p_j'})$ in Corollary 4.21 is optimal; when $\alpha_j = k(1-\frac{a}{p_j'})$, note that there may exist A_p weight w_j with $w_j' = w_j^{-1/(p_j-1)} \in RD_a(\mathbb{R}^k)$ such that $G_{\alpha_j,k} \notin L^{p_j'}_{w_j'}(\mathbb{R}^k)$. For example, one may take $w_j = 1$, then $a = 1$ and $G_{k/p_j,k} \notin L^{p_j'}(\mathbb{R}^k)$.

(c) In particular, if $\tilde{p} = (p,p)$, we have

$$\mathcal{L}^p_{w,\alpha_1+\alpha_2}(\mathbb{R}^n) \subset \mathcal{L}^{\tilde{p}}_{w,(\alpha_1,\alpha_2-\varepsilon),(n-k,k)}(\mathbb{R}^n) \xrightarrow{R} \mathcal{L}^p_{w_1,\alpha_1}(\mathbb{R}^{n-k})$$

if $\alpha_2 > k(1-\frac{a}{p'})$ and $\varepsilon > 0$ is chosen so that $\alpha_2 - \varepsilon > k(1-\frac{a}{p'})$ when $w_2' \in RD_a(\mathbb{R}^k)$. Thus $\mathcal{L}^p_{w,\alpha}(\mathbb{R}^n) \xrightarrow{R} \mathcal{L}^p_{w_1,\beta}(\mathbb{R}^{n-k})$ when $\alpha - \beta > k(1-\frac{a}{p'})$. Moreover, note that when $w_i = 1$, for $i = 1, 2$, then $a = 1$ and hence

$$\mathcal{L}^p_\alpha(\mathbb{R}^n) \xrightarrow{R} \mathcal{L}^p_\beta(\mathbb{R}^{n-k}) \tag{4.23}$$

if $\alpha - \beta > \frac{k}{p}$. Unfortunately, we are unable to conclude that

$$\mathcal{L}^p_\alpha(\mathbb{R}^n) \xrightarrow{R} \mathcal{L}^p_{\alpha-\frac{k}{p}}(\mathbb{R}^{n-k}). \tag{4.24}$$

However (4.24) is true when $1 < p \leq 2$ by [1, Theorem 11]. Nevertheless, we show that (4.23) is true for a larger space, i.e.,

$$\mathcal{L}^{\bar{p}}_{(\beta,\alpha-\beta),(n-k,k)}(\mathbb{R}^n) \xrightarrow{R} \mathcal{L}^p_\beta(\mathbb{R}^{n-k}).$$

Moreover, it is easy to see that if $f \in \mathcal{L}^{p_1}_{w_1,\beta}(\mathbb{R}^{n-k})$, then there exists an extension Ef of f such that $Ef \in \mathcal{L}^{(p_1,p_2)}_{w_1\times w_2,(\alpha,\beta)}(\mathbb{R}^n)$ for any $\alpha > 0, p_2 > 1$ and $w_2 \in A_{p_2}(\mathbb{R}^k)$. For example, we can take $Ef(x,y) = \eta(y)f(x)$, $x \in \mathbb{R}^{n-k}$, $y \in \mathbb{R}^k$, where $\eta \in C_0^\infty(\mathbb{R}^k)$ with $\eta(0) = 1$.

References

[1] Alberto P. Calderón, Lebesgue spaces of differentiable functions and distributions, Proc. Symp. Pure Math. Vol. IV (1961), 33-49.

[2] Seng-Kee Chua, Extension theorems on weighted Sobolev spaces, to appear.

[3] Seng-Kee Chua, Extension and restriction theorems on weighted Sobolev spaces, PhD thesis, Rutgers University, 1991.

[4] Ronald R. Coifman and Charles Fefferman, Weighted norm inequalities for maximal functions and singular integrals, Studia Math. 51 (1974), 241-250.

[5] Vladimir G. Maz'ja, Sobolev Spaces, Springer-Verlag, New York 1985.

[6] Nicholas Miller, Weighted Sobolev spaces and pseudodifferential operators with smooth symbols, Trans. Amer. Math. Soc. 269 (1982), no.1, 91-109.

[7] Benjamin Muckenhoupt, Weighted norm inequalities for the Hardy maximal function, Trans. Amer. Math. Soc. 165 (1972), 207-226.

[8] Esko Nieminen, Housdorff measures, capacities and Sobolev spaces with weights, Ann. Acad. Sci. Fenn. Series A. I. Dissertationes 81(1991).

[9] Elias M. Stein, Singular Integrals and Differentiability Properties of Functions, Princeton University Press, Princeton 1970.

[10] Elias M. Stein, The characterization of functions arising as potentials I, Bull. Amer. Math. Soc. 67 (1961), 102-104; II ibid., 68 (1962), 577-582.

[11] Alberto Torchinsky, Real-Variable Methods in Harmonic Analysis, Academic Press Inc., New York 1986.

[12] Richard Wheeden and Antoni Zygmund, Measure and Integral, Marcel Dekker Inc., New York 1977.

[13] Eric Sawyer and Richard L. Wheeden, Weighted inequalities for fractional integrals on Euclidean and homogeneous spaces, Amer. J. Math., to appear.

Received July 22, 1991