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Riemann Integral vs. Lebesgue Integral

Introduction. What more could one say about this old and basic subject? It is probably hard to answer such a question. Still, the purpose of this paper is to try to interpret a well-known parallelism:

$$\begin{array}{cc}
\text{Lebesgue integral} & \text{Riemann integral} \\
\updownarrow & \updownarrow \\
\text{countably additive measure} & \text{finitely additive measure} \\
\updownarrow & \updownarrow \\
\text{monotone convergence} & \text{uniform convergence}
\end{array}$$

One part of the preceding comparison (the one which deals with convergence) is expressed in the following two theorems, which are certainly familiar to the reader:

Theorem A. If a sequence $\{f_n\}$ of Riemann integrable functions on a compact interval $[a, b]$ converges uniformly to the function f , then f is Riemann integrable and

$$\lim_{n\to\infty} \int f_n = \int f .$$

Theorem B. If a nondecreasing sequence $\{f_n\}$ of nonnegative Lebesgue integrable functions converges pointwise to the function f and $\lim \int f_n$ is finite, then f is Lebesgue integrable and

$$\lim_{n\to\infty} \int f_n = \int f .$$

But, if we consider the definitions of these two integrals then this parallelism is somewhat obscured. To define the Lebesgue integral, we can start with a countably additive measure, define the integral for simple functions, and then extend it, by monotone convergence, to integrable functions. On the other hand, if we start with a finitely additive measure, then we have to deal with the sets of Lebesgue measure zero, and the procedure does not look so elegant and simple any more.

Therefore, the following question is "natural" to ask:

Is it possible to follow the pattern of the definition of the Lebesgue integral, but to use a finitely additive measure instead of a countably additive measure, and uniform convergence instead of monotone convergence, to obtain the Riemann integral so that the set of integrable functions is exactly the set of *all* Riemann integrable functions ?

The answer is *yes*. In the first section we will explain the defining procedure precisely (which is rather obvious, once we know what we want; we mention it for the sake of precision and completeness). In the second paragraph we will show that the Riemann integral can be obtained in that way.

While presenting this paper to Prof. Murali Rao, and discussing it with him, the second proof of the same theorem emerged. Since it gives additional insight into Riemann integrable functions, it will be presented in the third section. The reader will notice that the second proof is shorter and more elegant than the first one. It is also logically independent of the first proof. However the second proof illuminates only one side of the problem, namely, the description of the inverse image of an interval with respect to the Riemann integrable function. For a deeper understanding it is important to know if we can control the behaviour of a Riemann integrable function on the set of discontinuity points. The first proof emphasizes this aspect of the problem. I would like to take this opportunity to thank Prof. Murali Rao for the fruitful discussion, and for his generosity in allowing me to present the result of that discussion in this paper.

Remark 1 The definition of the Riemann integral given in this paper is, of course, not intended for a person not so familiar with the Riemann integral. □

1. The Definition of Integral. Let Ω be a nonempty set and $\mathcal{A}$ an algebra on Ω, i.e., $\mathcal{A}$ is a family of subsets of Ω which contains Ω, and is closed under complementation and finite unions.

Let $\mathbf{B} = \mathbf{B}(\Omega)$ be the set of all bounded real-valued functions on Ω. Notice that we do not require any other conditions (i.e., no measurability, continuity, etc.), except boundedness, for elements in $\mathbf{B}$. Together with the norm $\| f \| = \sup\{| f(\omega) | : \omega \in \Omega \}$ $\mathbf{B}$ is a Banach space, and convergence in norm $\| \ \|$ is exactly uniform convergence.

As usual, we will denote the set of natural numbers by $\mathbf{N}$, the set of rational numbers by $\mathbf{Q}$, and the set of real numbers by $\mathbf{R}$.

Let $\mathbf{S} = \mathbf{S}(\Omega, \mathcal{A})$ denote the set of all $\mathcal{A}$-simple functions on Ω, i.e., the set of all $f : \Omega \longrightarrow \mathbf{R}$ such that there exist $n \in \mathbf{N}$, $\alpha_1, \ldots, \alpha_n \in \mathbf{R}$, and $A_1, \ldots, A_n \in \mathcal{A}$ mutually disjoint, and such that

$$f = \sum_{i=1}^{n} \alpha_i \chi_{A_i} \quad ,$$

where χ_{A_i} is the characteristic function of the set A_i. Since $\mathcal{A}$ is an algebra, it follows that $\mathbf{S}$ is a vector space and, in fact, it is a subspace of $\mathbf{B}$. Let us denote by $\mathbf{I} = \mathbf{I}(\Omega, \mathcal{A})$ the closure of $\mathbf{S}$ in $(\mathbf{B}, \| \ \|)$. Then $(\mathbf{I}, \| \ \|)$ is a Banach space and $\mathbf{S}$ is a dense subspace of $\mathbf{I}$.

Let μ be a finite, finitely additive measure on $(\Omega, \mathcal{A})$, i.e., $\mu : \mathcal{A} \longrightarrow \mathbf{R}$ is such that, for every $A \in \mathcal{A}$, $0 \leq \mu(A) \leq \mu(\Omega) < +\infty$ and

$$\mu\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mu(A_i) \quad ,$$

whenever $A_i \in \mathcal{A}$ are mutually disjoint.

Let us define the integral with respect to μ . We define it first on the set of simple functions. If $f \in \mathbf{S}$ then there exist $n \in \mathbf{N}$, $\alpha_1, \ldots, \alpha_n \in \mathbf{R}$, and $A_1, \ldots, A_n \in \mathcal{A}$ mutually disjoint such that $f = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$. For such f we define the integral of f by

$$\mathrm{Int}(f) = \sum_{i=1}^{n} \alpha_i \mu(A_i) \quad . \tag{1}$$

It is now standard to show that $\mathrm{Int} : \mathbf{S} \longrightarrow \mathbf{R}$ is a well-defined linear functional on $\mathbf{S}$, and that moreover Int has the following properties:

(a) $(\forall\, \alpha, \beta \in \mathbf{R})$ $(\forall\, f, g \in \mathbf{S})$ $\quad \mathrm{Int}(\alpha f + \beta g) = \alpha \mathrm{Int}(f) + \beta \mathrm{Int}(g)$

(b) $(\forall\, A \,\epsilon \mathcal{A})$ $\quad \text{Int}(\chi_A) = \mu(A)$

(c) $(\forall f, g \,\epsilon \mathbf{S},\ f \leq g)$ $\quad \text{Int}(f) \leq \text{Int}(g)$

(d) $(\forall\, f \,\epsilon \mathbf{S})$ $\quad |\text{Int}(f)| \leq \mu(\Omega) \cdot \|f\|$.

It follows, since $\mu(\Omega)$ is finite, that Int is a bounded, positive, linear (and therefore also continuous) functional on the normed space $(\mathbf{S}, \|\ \|)$. Since $\mathbf{S}$ is dense in $\mathbf{I}$, Int can be uniquely (by continuity) extended to $\mathbf{I}$. By this extension we obtain $\text{Int} : \mathbf{I} \longrightarrow \mathbf{R}$, a bounded, positive, linear functional on the Banach space $(\mathbf{I}, \|\ \|)$, which satisfies all the properties (a), (b), (c), and (d), with $\mathbf{I}$ instead of $\mathbf{S}$ everywhere. It also satisfies the property

$$(e) \qquad \text{if } f_n \xrightarrow{\|\ \|} f,\ \{f_n\} \subseteq \mathbf{I},\ f \,\epsilon \mathbf{B} \quad \Longrightarrow \quad f \,\epsilon \mathbf{I} \text{ and}$$

$$\text{Int}(f_n) \longrightarrow \text{Int}(f) \quad .$$

Definition. We say that $f : \Omega \to \mathbf{R}$ is $\mu - integrable$ if $f \,\epsilon\, \mathbf{I}$. The μ-integral of f, which we denote by $\int f \, d\mu$, is defined to be $\text{Int}(f)$.

Remark 2 Notice that the μ-integral is defined exactly in the way requested in the introduction. We defined it first for $\mathcal{A}$-simple functions and then extended it, by uniform convergence, to the general case.□

One can think that the "natural" choice for the algebra $\mathcal{A}$ would be

$$\mathcal{J} = \{A \subseteq [a, b] \mid A \text{ is the finite disjoint union of } 1 - \text{intervals}\} \qquad (2)$$

where by 1-interval we mean any open, closed, or half-open subinterval of $[a, b]$. Certainly, $\mathcal{J}$ is an algebra, and every $\mathcal{J}$-simple function is Riemann integrable. Hence, by Theorem A., every $f \,\epsilon\, \mathbf{I}([a, b], \mathcal{J})$ is Riemann integrable, and $\mathbf{I}([a, b], \mathcal{J})$ contains all continuous functions. But, does it contain all the Riemann integrable functions? The following example shows that the answer is *no.* In the second section we will show which algebra does satisfy all the requirements.

Example 1 Let C denote the Cantor ternary set in [0,1]. Take h to be the characteristic function of C. Then h is continuous at every point in the complement of C, since C is closed. Hence, h is Riemann integrable, since the Lebesgue measure of C is zero. Suppose now that $h \,\epsilon\, \mathbf{I}([0, 1], \mathcal{J})$. Then there exists $f \,\epsilon\, \mathbf{S}([0, 1], \mathcal{J})$ such that $\|\, f - h \,\| < 1/4$. Since f is an $\mathcal{J}$-simple function, there exists a 1-interval A on which f is constant, say α, and such that A contains at least two points in C. In particular, the interior of A must be nonempty (otherwise A would be a 1-interval of the form [x,x]). But then A must intersect also the complement of C. Therefore, $|\alpha - 0| < 1/4$ and $|\alpha - 1| < 1/4$ which is not possible. Thus h is not in $\mathbf{I}([0,1], \mathcal{J})$. $\diamond$

2.Riemann Integral as μ-integral. Let us show that we can interpret the standard Riemann integral as a μ-integral. Let $\Omega = [a,b]$ be a closed interval, where $a, b \in \mathbf{R}$, $a < b$. We will denote by $int(A)$, $Cl(A)$, $Fr(A) = Cl(A) \cap Cl(A^c)$, the interior, the closure, and the frontier of the set A, respectively. For any function $f : [a,b] \to \mathbf{R}$, we will denote by $D(f)$ the set of discontinuity points of f, and by $C(f)$ its complement, the set of the points of continuity. Recall that, for every $A \subseteq [a,b]$,

$$D(\chi_A) = Fr(A) \quad . \tag{3}$$

Our choice of an algebra on $[a,b]$ is $\mathcal{F}$ defined by

$$\mathcal{F} = \{A \subseteq [a,b] \mid A \text{ is Lebesgue measurable and } \Lambda(Fr(A)) = 0\} \tag{4}$$

where Λ is Lebesgue measure.

The facts that Lebesgue measurable sets form a σ-algebra, and that $Fr(\emptyset) = \emptyset$, $Fr(A) = Fr(A^c)$, $Fr(A \cup B) \subseteq Fr(A) \cup Fr(B)$, imply that $\mathcal{F}$ is an algebra of subsets of $[a,b]$. Notice that $\mathcal{J} \subseteq \mathcal{F}$, $\mathcal{J} \neq \mathcal{F}$, where $\mathcal{J}$ is defined by (2). Notice also that one-point sets are in $\mathcal{F}$, but the set $\mathbf{Q} \cap [a,b]$ is not in $\mathcal{F}$. Therefore, $\mathcal{F}$ is not a σ-algebra. The following simple theorem better explains the reason why we took $\mathcal{F}$ into consideration.

Theorem 1 *A real function f on $[a,b]$ assuming only finitely many values (and therefore bounded) is Riemann integrable if and only if $f \in \mathbf{S}([a,b], \mathcal{F})$.*

Proof. If $f \in \mathbf{S}([a,b], \mathcal{F})$ then, by (3) and (4), it is a linear combination of Riemann integrable functions. Therefore, it is Riemann integrable, too.

If f is a finitely valued Riemann integrable function, then there exists a set of different nonzero real numbers $\{\alpha_1, \ldots, \alpha_n\}$ such that $f = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$, where $A_i = f^{-1}(\{\alpha_i\})$. Let us denote $f^{-1}(\{0\})$ by A_0. Since every Riemann integrable function is Lebesgue integrable, it follows that A_0, $A_1, \ldots, A_n$ are

Lebesgue measurable. Also, since the interval $[a, b]$ is equal to the disjoint union $A_0 \cup A_1 \cup \ldots \cup A_n$, and

$$C(f) = \bigcup_{i=0}^{n} int(A_i) \quad ,$$

it follows that

$$D(f) = \bigcap_{i=0}^{n} Cl(A_i^c) = Cl(A_0^c) \cap [\bigcap_{i=1}^{n} Cl(A_i^c)] =$$

$$= Cl(A_1 \cup \ldots \cup A_n) \cap [\bigcap_{i=1}^{n} Cl(A_i^c)] = \bigcup_{i=1}^{n} [Cl(A_i) \cap Cl(A_i^c)] = \bigcup_{i=1}^{n} Fr(A_i) \ .$$

Hence $\Lambda(D(f)) = 0$ implies that $\Lambda(Fr(A_i)) = 0$, for every $i = 1, \ldots, n$. Thus $f \in \mathbf{S}([a, b], \mathcal{F})$.

Q.E.D.

Remark 3 One may ask whether it is possible to replace $\mathcal{F}$ by another algebra $\mathcal{A}$ on $[a, b]$ which yields the following parallel with the Lebesgue integral.
(i) $\{x : f(x) > c\} \in \mathcal{A}$, for every Riemann integrable f, and for every real c, and
(ii) χ_A is Riemann integrable, for every A in $\mathcal{A}$.
The well-known example of "the denominator function" d (which is Riemann integrable) defined by

$$d(x) = \begin{cases} 1/n & \text{if } x = m/n \text{ and gr. com. divisor } (m, n) = 1 \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

shows that in that case $\mathbf{Q} \cap [a, b] = \{x : d(x) > 0\}$ must be in $\mathcal{A}$. But it is not possible, since $\chi_{\mathbf{Q} \cap [a,b]}$ is not Riemann integrable. Hence, we can not satisfy (i) and (ii), which shows that $\mathcal{F}$ gives the best description of simple Riemann integrable functions.
More detailed analysis of the inverse images of Riemann integrable functions will be given in the third paragraph.□

Let us proceed now to the main result of this paper. In the following text we will denote by Int the μ-integral, where μ is the restriction of the Lebesgue measure Λ to $([a, b], \mathcal{F})$.

Theorem 2 *The function $f : [a,b] \to \mathbf{R}$ is Riemann integrable if and only if $f \in \mathbf{I}([a,b],\mathcal{F})$, and, in this case,*

$$\int_a^b f(x)dx = \mathrm{Int}(f) \quad .$$

Proof. If $f \in \mathbf{I}([a,b],\mathcal{F})$, then there is a sequence of simple functions $\{f_n\} \subseteq \mathbf{S}([a,b],\mathcal{F})$, which converges to f uniformly. By Theorem 1., every f_n is a Riemann integrable function and, by the definition of Int,

$$\int_a^b f_n(x)dx = \mathrm{Int}(f_n) \quad .$$

Now, using Theorem A. and property (e) of Int, we obtain that f is Riemann integrable and that the desired equality of integrals is correct.
Therefore, it remains only to prove that every Riemann integrable function $f : [a,b] \to \mathbf{R}$ belongs to $\mathbf{I} = \mathbf{I}([a,b],\mathcal{F})$.

If f is Riemann integrable, then $\Lambda(D(f)) = 0$ and there exists $R > 0$ such that $|f(x)| \leq R$, for every x in $[a,b]$. In particular, $C(f)$ is dense in $[a,b]$. Let $\varepsilon > 0$ be a positive real number. We would like to prove that there exists $g \in \mathbf{S}([a,b],\mathcal{F})$ such that

$$\| f - g \| \leq \varepsilon \quad .$$

For every $x \in C(f)$ there exists $\delta_x > 0$ such that $|x - y| < \delta_x$ implies $|f(x) - f(y)| < \varepsilon/3$. Consider now two sets defined by

$$O_\varepsilon = \bigcup_{x \in C(f)} (x - \delta_x,\, x + \delta_x) \quad \text{and} \quad K_\varepsilon = O_\varepsilon^c \quad . \tag{6}$$

O_ε is an open, dense set and K_ε is a compact set of Lebesgue measure zero. Hence O_ε, $K_\varepsilon \in \mathcal{F}$, since $Fr(O_\varepsilon) = Fr(K_\varepsilon) = K_\varepsilon$. In particular, the functions

$$f_1 = f \cdot \chi_{O_\varepsilon} \quad \text{and} \quad f_2 = f \cdot \chi_{K_\varepsilon} \tag{7}$$

are Riemann integrable and $f = f_1 + f_2$. Therefore it is enough to approximate f_1 and f_2 by $\mathcal{F}$-simple functions.
Consider f_2 first. We partition the interval $[-R,R]$ into disjoint subintervals $B_0 = [-R, -R + \varepsilon/3]$, $B_1 = (-R + \varepsilon/3, -R + \varepsilon 2/3], \ldots,$ $B_k = (-R + \varepsilon k/3,\, R]$, where k is the smallest positive integer such that $(k+1)\varepsilon/3 > 2R$. For every $i = 0, 1, \ldots, k$, we define $A_i = f_2^{-1}(B_i)$. Then

the sets A_i are Lebesgue measurable, since f_2 is Riemann (and therefore Lebesgue) integrable. There is exactly one $i_0 \in \{0, 1, \ldots, k\}$ such that $0 \in B_{i_0}$. Hence, $A_i \subseteq K_\varepsilon$ and $Fr(A_i) \subseteq K_\varepsilon$, for every $i \neq i_0$. It shows that $A_i \in \mathcal{F}$, for every $i \neq i_0$. Then all A_i are in $\mathcal{F}$, since they form a partition of the interval $[a, b]$. It follows then that the function g_2 defined by

$$g_2 = \sum_{i=0}^{k} (-R + \frac{i\varepsilon}{3}) \cdot \chi_{A_i} \tag{8}$$

belongs to $\mathbf{S}([a, b], \mathcal{F})$ and, by the definition of g_2, that

$$\| f_2 - g_2 \| \leq \frac{\varepsilon}{3} \quad . \tag{9}$$

Consider f_1. Since $[a, b]$ is a separable metric space there exists a sequence $\{x_n\} \subseteq C(f)$ such that $O_\varepsilon = \bigcup_{n=1}^{\infty} O_n$, where $O_n = (x_n - \delta_{x_n} , x_n+ +\delta_{x_n}) \in \mathcal{J} \subseteq \mathcal{F}$, for every $n \geq 1$. We define inductively $W_1 = O_1$ and $W_{n+1} = O_{n+1} \backslash (O_1 \cup \ldots \cup O_n)$, a disjoint family of sets in $\mathcal{J} \subseteq \mathcal{F}$ (since these are algebras), such that its union is O_ε. We define the function h_1 by

$$h_1 = \sum_{n=1}^{\infty} f_1(x_n) \cdot \chi_{W_n} \quad . \tag{10}$$

Notice that $\Lambda(int(W_n)) = \Lambda(W_n)$, since every W_n is the finite disjoint union of 1-intervals. Then the computation

$$b - a = \Lambda(O_\varepsilon) = \sum_{n=1}^{\infty} \Lambda(W_n) = \sum_{n=1}^{\infty} \Lambda(int(W_n)) \leq \Lambda(C(h_1))$$

shows that $\Lambda(D(h_1)) = 0$. Together with the fact that h_1 is bounded, it implies that h_1 is Riemann integrable and

$$\| f_1 - h_1 \| \leq \frac{\varepsilon}{3} \quad . \tag{11}$$

The function h_1 is countably valued, but need not be simple. There exist, since the W_n's are disjoint, a sequence $\{V_n\}$ of mutually disjoint 1-intervals whose union is equal to O_ε, and a sequence $\{\alpha_n\}$ of real numbers, such that

$$h_1 = \sum_{n=1}^{\infty} \alpha_n \cdot \chi_{V_n} \quad .$$

Let us denote the endpoints of V_n by c_n and d_n, where $c_n \leq d_n$. Then $b - a = \sum_{n=1}^{\infty}(d_n - c_n)$ shows that $G = \bigcup_{n=1}^{\infty}(c_n, d_n) \subseteq O_\varepsilon$ is an open, dense set such that $Fr(G) = G^c$ and $\Lambda(G^c) = 0$. Notice also that G^c contains all the points c_n and d_n, since $(c_n, d_n) \subseteq V_n$ are mutually disjoint. It follows that

$$\Lambda\left[Cl(\{c_n : \ n \geq 1\} \cup \{d_n : \ n \geq 1\})\right] = 0 \ . \tag{12}$$

For every $S \subseteq \mathbf{N}$ we define $V_S = \bigcup_{i \epsilon S} V_i$. Then every V_S is Lebesgue measurable and $V_S^c = V_{S^c} \cup K_\varepsilon$. The computation

$$Fr(V_S) = Cl(V_S) \cap [Cl(K_\varepsilon) \cup Cl(V_{S^c})] =$$

$$= [Cl(V_S) \cap Cl(K_\varepsilon)] \cup [Cl(V_S) \cap Cl(V_{S^c})] \subseteq$$

$$\subseteq K_\varepsilon \cup \left[Cl(V_S) \backslash \left(\bigcup_{i \epsilon S}(c_i, d_i)\right)\right] \subseteq K_\varepsilon \cup Cl\left(\{c_n : n \geq 1\} \cup \{d_n : n \geq 1\}\right)$$

shows, together with (12), that

$$V_S \,\epsilon\, \mathcal{F} \quad , \text{ for every } S \subseteq \mathbf{N} \ . \tag{13}$$

We consider now the same partition $\{B_i\}$ of the interval $[-R, R]$, as we did before. For every $i = 0, 1, \ldots, k$, we define $S_i = \{m \,\epsilon\, \mathbf{N} \mid \alpha_m \,\epsilon\, B_i\}$. Then $V_{S_i} = h_1^{-1}(B_i)$, and (13) implies that the function

$$g_1 = \sum_{i=0}^{k}(-R + \frac{i\varepsilon}{3}) \cdot \chi_{V_{S_i}} \tag{14}$$

belongs to $\mathbf{S}([a, b], \mathcal{F})$. The definition (14) implies that

$$\| \, g_1 - h_1 \, \| \leq \frac{\varepsilon}{3} \ . \tag{15}$$

Finally, we combine (7), (9), (11), and (15) to obtain

$$\| \, f - (g_1 + g_2) \, \| \leq \| \, f_1 - g_1 \, \| + \| \, f_2 - g_2 \, \| \leq \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \ . \tag{16}$$

Since $\varepsilon > 0$ was arbitrary, it follows that $f \,\epsilon\, \mathbf{I}([a, b], \mathcal{F})$.

Q.E.D.

3.More on Riemann Integrable Functions. The proof of Theorem 2. is so lengthy because, for a Riemann integrable function $f : [a, b] \to \mathbf{R}$ and a 1-interval $B \subseteq \mathbf{R}$ with endpoints c and d, $f^{-1}(B)$ need not belong to the algebra $\mathcal{F}$. Still, another way of analyzing this problem is possible, which gives more information on the sets of the form $f^{-1}(B)$.

Consider a point $x \in C(f) \cap Fr(f^{-1}(B)) = C(f) \cap Cl(f^{-1}(B)) \cap \cap Cl(f^{-1}(B^c))$. Then there exist two sequences $\{y_n\} \subseteq f^{-1}(B)$ and $\{z_n\} \subseteq f^{-1}(B^c)$, such that $y_n \to x$ and $z_n \to x$. Since $x \in C(f)$, it follows that $f(y_n) \to f(x)$ and $f(z_n) \to f(x)$. But $f(y_n) \in B$ and $f(z_n) \in B^c$, and B being a 1-interval, imply that $f(x) = c$ or $f(x) = d$. This shows that, for every 1-interval $B \subseteq \mathbf{R}$, with endpoints c and d, and for every Riemann integrable function $f : [a, b] \to \mathbf{R}$

$$Fr(f^{-1}(B)) \subseteq D(f) \cup f^{-1}(\{c\}) \cup f^{-1}(\{d\}) \quad . \tag{17}$$

Thus, it follows that $(\Lambda[f^{-1}(\{c\})] = \Lambda[f^{-1}(\{d\})] = 0 \implies f^{-1}(B) \in \mathcal{F})$, since $\Lambda(D(f)) = 0$. Of course, the example of "the denominator function" d shows that $\Lambda[f^{-1}(\{c\})]$ can be strictly positive. But still, if it happens that the set M, defined by

$$M = \{c \in \mathbf{R} \mid \Lambda[f^{-1}(\{c\})] > 0\} \quad , \tag{18}$$

is "small", then we can prove Theorem 2. in a simpler way.

More precisely, if $\Lambda(M) = 0$, then, for every $\varepsilon > 0$, there is a partition $t_0 = -R < t_1 < \ldots < t_n = R$ of the interval $[-R, R]$ (for the notation see the proof of Theorem 2), such that $t_i - t_{i-1} < \varepsilon$, and such that $t_i \in M^c$, for all i. Then it follows, by (17), that all the sets of the form $f^{-1}((t_{i-1}, t_i])$ belong to $\mathcal{F}$. We define the function g in $\mathbf{S}([a, b], \mathcal{F})$ simply by

$$g = t_0 \cdot \chi_{f^{-1}([t_0,t_1])} + t_1 \cdot \chi_{f^{-1}((t_1,t_2])} + \cdots + t_{n-1} \cdot \chi_{f^{-1}((t_{n-1},t_n])} \quad .$$

Obviously $\| f - g \| \leq \varepsilon$, which would then complete the proof of the Theorem 2.

The question is now, is the set M "small" enough. The following theorem shows that even more is true, i.e., that M is at most countable. The example after the theorem shows that M can be infinite. These two facts characterize M completely, and with that we will finish this paper.

Theorem 3 *If $f : [a, b] \to \mathbf{R}$ is Riemann integrable, then the set M, defined by (18), is at most countable.*

Proof. Notice that $M = \bigcup_{n=1}^{\infty} M_n$, where

$$M_n = \{c \in \mathbf{R} \mid \Lambda[f^{-1}(\{c\})] > \frac{1}{n}\} \ . \tag{19}$$

We claim that each M_n is finite. If M_n would be infinite then the interval $[a, b]$ would contain infinitely many sets of the form $f^{-1}(\{c\})$, and each of them has the Lebesgue measure bigger than $1/n$. But, if $c \neq d$ then $f^{-1}(\{c\}) \cap f^{-1}(\{d\}) = \emptyset$, which implies that $\Lambda[a, b] = +\infty$. This is an obvious contradiction.

Q.E.D.

Example 2 Let d be "the denominator function" defined by (5), and let $\{b_n\}$ be a strictly decreasing sequence in (a, b) which converges to a. We denote b by b_0, and define f by

$$f(x) = d(x) + \frac{1}{n} \text{ for } x \in (b_n, b_{n-1}] \text{ and } f(a) = 0 \ . \tag{20}$$

Then $f : [a, b] \to [0, 2]$ and $D(f) \subseteq (\mathbf{Q} \cap [a, b]) \cup \{a\} \cup \{b_n : \ n \geq 0\}$, which implies $\Lambda(D(f)) = 0$. Hence, f is Riemann integrable. Also, for every $n \geq 1$,

$$\Lambda[f^{-1}(\{\frac{1}{n}\})] \geq b_{n-1} - b_n > 0 \ ,$$

which shows that M is infinite.$\diamond$

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