Real Analysis Exchange Vol. 17 (1991–92)

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SOME REMARKS ON SUP-MEASURABILITY

Abstract. We give an example of a sup-measurable and non-measurable function in the general case associated with products of σ-ideals. We show that any behaviour of a function $F : \mathbf{R}^2 \to \mathbf{R}$ with respect to (L)- and (B)-measurability, and (L)- and (B)-sup-measurability is possible. Continuum Hypothesis is replaced by weaker conditions. Finally, the notion of quasi-sup-measurability is considered.

0. Introduction. A function $F : \mathbf{R}^2 \to \mathbf{R}$ (where $\mathbf{R}$ denotes the real line) is called *Lebesgue sup-measurable* (in short *(L)-sup-measurable*) if for each Lebesgue measurable function (abbr. *(L)-measurable*) $f : \mathbf{R} \to \mathbf{R}$, the function $g_{\langle F,f\rangle} : \mathbf{R} \to \mathbf{R}$ given by $g_{\langle F,f\rangle}(x) = F(x, f(x))$ is Lebesgue measurable (cf. [S], [GL], [G], [GG]). An analogous definition can be formulated when Lebesgue measurability is replaced by the possessing of the Baire property; we then use the phrases "*(B)-sup-measurable*" and "*(B)-measurable*" (cf. [GG]). The phrases "(L)-measurable" and "(B)-measurable" will also be used for sets in an obvious sense. It is easy to show that the (L)-measurability of $F : \mathbf{R}^2 \to \mathbf{R}$ need not imply its (L)-sup-measurability and the analogous implication for (B)-measurability and (B)-sup-measurability need not hold. It suffices to consider the characteristic function of the graph of any Borel function $f : \mathbf{R} \to \mathbf{R}$ (e.g. $f(x) = x$) restricted to a non-(L)-measurable (respectively, non-(B)-measurable) set in $\mathbf{R}$ (cf. [S], [G]). The converse implications are also false (see [GL], [GG]). The main idea of the proof contained in [GL] uses transfinite induction and supposes CH (Continuum Hypothesis). In Section 1 we observe that the scheme from [GL] can be extended to more general cases, and that CH can be replaced by another assumption which is implied by CH and turns out weaker than CH for (L)-measurability and (B)-measurablity. It seems an open problem whether the statement "(L)-sup-measurability need not imply (L)-measurability" is provable within ZFC. Professor Harazišvili has informed me that the result of [GL] was obtained by him independently and published in one of Georgian journals (see [H], Exercise 3, p.82 and also Exercise 4 where some interesting application is presented). Our generalization uses Fubini products of σ-ideals. Finally, we present the recent result of M. Penconek from Warsaw University.

In Section 2 we choose k arbitrary properties ($0 \leq k \leq 4$) out of the following: (L)-measurability, (B)-measurability, (L)-sup-measurability, (B)-sup-measurability. We prove the existence of an $F : \mathbf{R}^2 \to \mathbf{R}$ which has the chosen k properties and has not the remaining $4-k$ ones. For some cases, the scheme from [GL] is used again.

In Section 3 we introduce and study the notions of quasi-(L)-sup-measurability and quasi-(B)-sup-measurability.

1. A General Case of Sup-Measurability. Throughout the paper, if X is an uncountable set, we consider only these σ-ideals of subsets of X, different from the power set $\mathcal{P}(X)$, which either contain all singletons or are equal to $\{\emptyset\}$. The σ-ideal of all countable subsets of $\mathbf{R}$ will be denoted by $\mathcal{T}_0$.

From now on we fix two arbitrary σ-ideals $\mathcal{T}$ and $\mathcal{J}$ of subsets of $\mathbf{R}$. By $\mathcal{B}^{(2)}$, we denote the family of all Borel sets in $\mathbf{R}^2$. Define (cf. e.g. [CKP])

$$\mathcal{T} \otimes \mathcal{J} = \{E \subseteq \mathbf{R}^2 : (\exists\, B \in \mathcal{B}^{(2)})(E \subseteq \mathbf{R}\ \&\ \{x \in \mathbf{R} : B_x \notin \mathcal{J}\} \in \mathcal{T})\}$$

where $B_x = \{y \in \mathbf{R} : \langle x, y \rangle \in B\}$. Then $\mathcal{T} \otimes \mathcal{J}$ forms a σ-ideal of subsets of $\mathbf{R}^2$ which is called the *(Fubini) product* of $\mathcal{T}$ and $\mathcal{J}$. Define (see [F], p.16)

$$\operatorname{non}(\mathcal{T}) = \min\{|E| : E \subseteq \mathbf{R}\ \&\ E \notin \mathcal{T}\}$$

where $|E|$ stands for the cardinality of E. Observe that $\operatorname{non}(\mathcal{T}) \leq c$ (where $c = |\mathbf{R}|$). We have $\operatorname{non}(\{\emptyset\}) = 1$ and $\operatorname{non}(\mathcal{T}) \geq \omega_1$ for $\mathcal{T} \neq \{\emptyset\}$.

By $S(\mathcal{T})$ we denote the σ-algebra generated by all Borel sets and all sets from $\mathcal{T}$; that symbol is also used when we deal with a σ-ideal of sets in any topological space.

We say that $F : \mathbf{R}^2 \to R$ is $S(\mathcal{T})$*-sup-measurable* if, for each $S(\mathcal{T})$-measurable $f : \mathbf{R} \to \mathbf{R}$, the function $g_{\langle F,f \rangle}$ given by $g_{\langle F,f \rangle}(x) = F(x, f(x))$ is $S(\mathcal{T})$-measurable. Observe that all $S(\mathcal{T})$-sup-measurable functions form a linear space.

1.1. Lemma. *A function $F : \mathbf{R}^2 \to \mathbf{R}$ is $S(\mathcal{T})$-sup-measurable if and only if $g_{\langle F,f \rangle}$ is $S(\mathcal{T})$-measurable for each Borel $f : \mathbf{R} \to \mathbf{R}$.*

Proof. Necessity is self-evident. To show sufficiency, consider any $S(\mathcal{T})$-measurable $h : \mathbf{R} \to \mathbf{R}$. There is a Borel $f : \mathbf{R} \to \mathbf{R}$ such that $\{x \in \mathbf{R} : f(x) \neq h(x)\} \in \mathcal{T}$ (cf. [F], 1D(c)). Now, since $g_{\langle F,f \rangle}$ is $S(\mathcal{T})$-measurable, so is $g_{\langle F,h \rangle}$. ■

Now, let X be a separable and complete metric space. We say that $H \subseteq X$ is a *Bernstein set* if H intersects each perfect (nonempty) subset of X and $X \setminus H$ has the same property (for the construction of H, see e.g. [Kr], §40 I).

If each set of a σ-ideal $\mathcal{N} \subseteq \mathcal{P}(X)$ has a Borel superset from $\mathcal{N}$, we say that $\mathcal{N}$ has a *Borel base*.

The following lemma results immediately from [I]:

1.2. Lemma. *If a σ-ideal $\mathcal{N} \subseteq P(X)$ has a Borel base, then no Bernstein set is in $S(\mathcal{N})$.* ■

1.3. Lemma. *There is an $F : \mathbf{R}^2 \to \{0,1\}$ such that $F^{-1}[\{1\}] \in \{\emptyset\} \otimes \mathcal{T}_0$ and F is non-$S(\mathcal{T})$-sup-measurable for any $\mathcal{T}$ with a Borel base.*

Proof. Fix a Borel function $f : \mathbf{R} \to \mathbf{R}$ and a Bernstein set $H \subseteq \mathbf{R}$. Let $F : \mathbf{R}^2 \to \mathbf{R}$ be the characteristic function of the graph $Gr(f|H)$. Observe that $g_{\langle F,f\rangle}{}^{-1}[\{1\}] = H$, hence F is non-$S(\mathcal{T})$-sup-measurable by virtue of Lemma 1.2. The rest follows from $Gr(f|H) \subseteq Gr(f) \in \{\emptyset\} \otimes \mathcal{T}_0$. ■

Since $\{\emptyset\} \otimes \mathcal{T}_0 \subseteq \mathcal{T} \otimes \mathcal{J}$ when $\mathcal{T}_0 \subseteq \mathcal{J}$ (i.e. $\mathcal{J} \neq \{\emptyset\}$), we get

1.4. Corollary. *There is an $F : \mathbf{R}^2 \to \{0,1\}$ which is $S(\mathcal{T} \otimes \mathcal{J})$-measurable and non-$S(\mathcal{T})$-sup-measurable for any $\mathcal{T}$ with a Borel base and $\mathcal{J} \neq \{\emptyset\}$.* ■

The question arises when there exists an $F : \mathbf{R}^2 \to \mathbf{R}$ which is $S(\mathcal{T})$-sup-measurable and non-$S(\mathcal{T} \otimes \mathcal{J})$-measurable.

Let $pr : \mathbf{R}^2 \to \mathbf{R}$ denote the projection on the first coordinate.

1.5. Proposition. *The following conditions are equivalent:*

(1) *there is an $F : \mathbf{R}^2 \to \mathbf{R}$ which is $S(\mathcal{T})$-sup-measurable and non-$S(\mathcal{T} \otimes \mathcal{J})$-measurable,*

(2) *there is an $H \subseteq R^2$ such that $H \notin S(\mathcal{T} \otimes \mathcal{J})$ and $pr[H \cap Gr(f)] \in S(\mathcal{T})$ for each Borel $f : \mathbf{R} \to \mathbf{R}$,*

(3) *there is an $F : \mathbf{R}^2 \to \{0,1\}$ which is $S(\mathcal{T})$-sup-measurable and non-$S(\mathcal{T} \otimes \mathcal{J})$-measurable.*

Proof. (1) $\Rightarrow$ (2) There is $a \in \mathbf{R}$ such that $H = F^{-1}[(-\infty, a)] \notin S(\mathcal{T} \otimes \mathcal{J})$. For any Borel $f : \mathbf{R} \to \mathbf{R}$, we have

$$pr[H \cap Gr(f)] = \{x \in \mathbf{R} : F(x, f(x)) < a\} \in S(\mathcal{T})$$

since F is $S(\mathcal{T})$-sup-measurable.

(2) $\Rightarrow$ (3) Define F as the characteristic function of H. Obviously, F is non-$S(\mathcal{T} \otimes \mathcal{J})$-measurable. For any Borel $f : \mathbf{R} \to \mathbf{R}$, the function $g_{\langle F,f\rangle}$ maps $\mathbf{R}$ into $\{0,1\}$ and

$$(g_{\langle F,f\rangle})^{-1}[\{-1\}] = pr[H \cap Gr(f)] \in S(\mathcal{T}).$$

Thus F is $S(\mathcal{T})$-sup-measurable by Lemma 1.1.

(3) $\Rightarrow$ (1) is obvious. ■

Now, we are going to adapt the ideas of [GL] to show that (1) holds when $\mathrm{non}(\mathcal{T}) = c$ and $\mathcal{J} \neq \{\emptyset\}$.

1.6. Lemma. *If the graph $Gr(h)$ of a function $h : E \to \mathbf{R}$, $E \subseteq \mathbf{R}$, intersects each member of $\mathcal{B}^{(2)} \setminus (\mathcal{T}_0 \otimes \mathcal{T}_0)$, then $Gr(h) \notin S(\mathcal{T} \otimes \mathcal{J})$ for any $\mathcal{T} \neq \{\emptyset\}$ and $\mathcal{J} \neq \{\emptyset\}$.*

Proof. By the assumption, we have $\mathcal{T}_0 \subseteq \mathcal{T}$ and $\mathcal{T}_0 \subseteq \mathcal{J}$. Consequently, $\mathcal{T}_0 \otimes \mathcal{T}_0 \subseteq \mathcal{T} \otimes \mathcal{J}$ and thus, $\mathcal{B}^{(2)} \setminus (\mathcal{T} \otimes \mathcal{J}) \subseteq \mathcal{B}^{(2)} \setminus (\mathcal{T}_0 \otimes \mathcal{T}_0)$. Therefore $Gr(h)$ intersects each member of $\mathcal{B}^{(2)} \setminus (\mathcal{T} \otimes \mathcal{J})$. To get the assertion, let us first observe that $Gr(h) \notin \mathcal{T} \otimes \mathcal{J}$. Indeed, if $Gr(h) \in \mathcal{T} \otimes \mathcal{J}$, choose a Borel $D \in \mathcal{T} \otimes \mathcal{J}$ containing $Gr(h)$. We have $\mathbf{R}^2 \setminus D \in \mathcal{B}^{(2)} \setminus (\mathcal{T} \otimes \mathcal{J})$ hence $Gr(h)$ meets $\mathbf{R}^2 \setminus D$, a contradiction. Now, it suffices to exclude the case $Gr(h) \in S(\mathcal{T} \otimes \mathcal{J}) \setminus (\mathcal{T} \otimes \mathcal{J})$. If it holds, we get $Gr(h) = (B \setminus A) \cup (A \setminus B)$ for some $B \in \mathcal{B}^{(2)} \setminus (\mathcal{T} \otimes \mathcal{J})$ and $A \in \mathcal{T} \otimes \mathcal{J}$. Choose $C \in \mathcal{B}^{(2)}$ such that $A \subseteq C \in \mathcal{T} \otimes \mathcal{J}$. Then $B \setminus C \notin \mathcal{T} \otimes \mathcal{J}$. However, $(B \setminus C)_x \subseteq (Gr(h))_x \in \mathcal{J}$ for all $x \in \mathbf{R}$ which together with $B \setminus C \in \mathcal{B}^{(2)}$ gives $B \setminus C \in \mathcal{T} \otimes \mathcal{J}$, a contradiction. ■

1.7. Proposition. *There are $E \subseteq \mathbf{R}$ and $h : E \to \mathbf{R}$ such that*

(a) *$Gr(h)$ intersects each member of $\mathcal{B}^{(2)} \setminus (\mathcal{T}_0 \otimes \mathcal{T}_0)$,*

(b) *$|\{x \in E : h(x) = f(x)\}| < c$ for each Borel $f : \mathbf{R} \to \mathbf{R}$.*

Proof. Arrange all Borel functions from $\mathbf{R}$ into $\mathbf{R}$ in a $1-1$ sequence $\{f_\alpha\}_{\alpha<c}$ and all sets from $\mathcal{B}^{(2)} \setminus (\mathcal{T}_0 \otimes \mathcal{T}_0)$ - in a sequence $\{E_\alpha\}_{\alpha<c}$. Let us choose $\langle x_0, y_0\rangle \in E_0$. For any $\alpha < c$, having all $\langle x_\gamma, y_\gamma\rangle$, $\gamma < \alpha$, defined, let us choose

$$x_\alpha \in (\{x \in \mathbf{R} : (E_\alpha)_x \notin \mathcal{T}_0\} \setminus \{x_\gamma : \gamma < \alpha\}.$$

The choice is possible since $\{x \in \mathbf{R} : (E_\alpha)_x \notin \mathcal{T}_0\}$ is uncountable and analytic (see [Kr], p.496) hence is of cardinality c ([Kr], 39 I). Similarly, one can choose

$$y_\alpha \in (E_\alpha)_{x_\alpha} \setminus \{f_\gamma(x_\alpha) : \gamma < \alpha\}$$

since $(E_\alpha)_{x_\alpha}$ is Borel and uncountable hence of cardinality c. The induction gives us the sequence $\{\langle x_\alpha, y_\alpha\rangle\}_{\alpha<c}$. Put $E = \{x_\alpha : \alpha < c\}$ and $h(x_\alpha) = y_\alpha$ for $\alpha < c$.

Now, (a) follows immediately from the construction. To get (b) observe that, for any f_γ, $\gamma < c$, we have $\{x \in E : h(x) = f_\gamma(x)\} \subseteq \{x_\alpha : \alpha \leq \gamma\}$. ■

1.8. Corollary. *There is an* $F : \mathbf{R}^2 \to \{0, 1\}$ *such that*

(a) F *is* $S(\mathcal{T})$*-sup-measurable for each* $\mathcal{T}$ *with non*$(\mathcal{T}) = c$,

(b) F *is non-*$S(\mathcal{T} \otimes \mathcal{J})$*-measurable for any* $\mathcal{T} \neq \{\emptyset\}$ *and* $\mathcal{J} \neq \{\emptyset\}$.

Proof. Define F as the characteristic function of $Gr(h)$ where h is taken from 1.7. From non$(\mathcal{T}) = c$ and 1.7(b) we get assertion (a). Assertion (b) follows from 1.6 and 1.7(a). ■

Note that our notion of sup-measurability can be extended in an obvious way to the case of $F : X \times Y \to \mathbf{R}$ where X and Y are uncountable Polish spaces. Then the respective versions of 1.1, 1.4 and 1.8 are true. Speaking about applications of 1.8 notice that, besides the σ-ideal of all sets of size $< c$, we know at least two interesting examples of σ-ideals with non ( ) $= c$: the Marczewski σ-ideal of (s^0) sets ([W], Th. 2.1) and the Mycielski σ-ideal on 2^ω or ω^ω ([R], Th. 2.3).

Corollary 1.7 can also be applied to the cases $\mathcal{T} = \mathcal{J} = \mathbf{L}$ and $\mathcal{T} = \mathcal{J} = \mathbf{K}$ where $\mathbf{L}$ and $\mathbf{K}$ stand for the ideals of Lebesgue null sets and of meager sets (i.e. those of the first category) in $\mathbf{R}$, respectively. Following our notation from Section 0, instead of "$S(\mathbf{K})$-(sup-)measurable" and "$S(\mathbf{L})$-(sup-)measurable" we write "(B)-(sup-) measurable" and "(L)-(sup-)measurable". We get

1.9. Corollary. (a) *If non* $(\mathbf{L}) = c$, *there is an* $F : \mathbf{R}^2 \to \{0, 1\}$ *which is* (L)*-sup-measurable and non-*(L)*-measurable (cf. [GL], Th.2).*

(b) *If non* $(\mathbf{K}) = c$, *there is an* $F : \mathbf{R}^2 \to \{0, 1\}$ *which is* (B)*-sup-measurable and non-*(B)*-measurable (cf. [GG], Th.11).* ■

Note that CH was assumed in [GL] and [GG]. Our results are sharper since CH evidently implies non $(\mathbf{K}) =$ non $(\mathbf{L}) = c$ and there are models of ZFC in which $\omega_1 <$ non $(\mathbf{K}) = c$ and $\omega_1 <$ non $(\mathbf{L}) = c$ (see [M]). Also Martin's axiom implies that non $(\mathbf{K}) =$ non $(\mathbf{L}) = c$ (cf. [Kn]). Let us stress that, due to Proposition 1.8, the same F satisfies (a) and (b) of 1.9 simultaneously.

1.10. Problem. Can the existence of an (L)-sup-measurable and non-(L)-measurable (resp. (B)-sup-measurable and non-(B)-measurable) function be proved within ZFC?

Recently, some progress has been made by Marcin Penconek who proved the above-mentioned existence when non $(\mathbf{L}) = cf(c) < c$ (resp. non $(\mathbf{K}) = cf(c) < c$.)

1.11. Theorem (M. Penconek). (a) *If non* $(\mathbf{L}) = cf(c)$, *there exists a non-(L)-measurable function which is (L)-sup-measurable.*

(b) *If non*$(\mathbf{K}) = cf(c)$, *there exists a non-(B)-measurable function which is (B)-sup-measurable.*

Proof. We shall show (a); the proof of (b) is analogous. Let $\kappa = cf(c)$. If $\kappa = c$, we apply 1.9. So, assume that $\kappa < c$. Since non $(\mathbf{L}) = \kappa$, there is a non-(L)-measurable set $X = \{x_\alpha : \alpha < \kappa\}$ in $\mathbf{R}$. Let $\{B_\alpha\}_{\alpha<\kappa}$ be an increasing sequence of subsets of c (we identify c with the set of its predecessors), such that $|B_\alpha| < c$ and $\bigcup_{\alpha<\kappa} B_\alpha = c$. Consider a fixed enumeration $\{f_\alpha\}_{\alpha<c}$ of all Borel functions from $\mathbf{R}$ to $\mathbf{R}$. According to 1.5, it suffices to find a non-(L)-measurable $A \subseteq \mathbf{R}^2$ such that $pr[A \cap Gr(f)]$ is (L)-measurable for each Borel $f : \mathbf{R} \to \mathbf{R}$. By λ_2 we denote the plane Lebesgue measure. We define

$$A = \bigcup_{\alpha<\kappa} ((\{x_\alpha\} \times \mathbf{R}) \setminus \bigcup_{\nu\in B_\alpha} \{\langle x_\alpha, f_\nu(x_\alpha)\rangle\}).$$

Observe that the inner plane measure of A is zero. Indeed, if it is not the case, there exists a closed $B \subseteq A$ with $\lambda_2(B) > 0$ and thus $B^* = \{x \in \mathbf{R} : B_x \notin \mathbf{L}\}$ is (L)-measurable of positive measure. Hence $|B^*| = c$. Now, for $x \in B^* \setminus X$, we have $A_x = \emptyset$ (by the definition of A) and $A_x \neq \emptyset$ (since $\emptyset \neq B_x \subseteq A_x$), a contradiction. Suppose that $\lambda_2(A) = 0$. Then, by the Fubini theorem, $A_x \in \mathbf{L}$ for each x from a set E of full measure. Obviously, E meets X and, for $x \in X \cap E$, $x = x_\alpha$, we have $|\mathbf{R} \setminus A_x| = |\bigcup_{\nu\in B_\alpha}\{f_\nu(x)\}| < c$, which is impossible because $A_x \in \mathbf{L}$ implies $|\mathbf{R} \setminus A_x| = c$. Since inner and outer measures of A differ, A is non-(L)-measurable. To end the proof, fix any Borel function $f = f_\nu$. There is an α_0 such that $\nu \in B_\alpha$ for all α, $\alpha_0 \leq \alpha < \kappa$. Thus $\langle x_\alpha, f(x_\alpha)\rangle \notin A$ for $\alpha \geq \alpha_0$ and, consequently, $pr[A \cap Gr(f)] \subseteq \bigcup_{\alpha<\alpha_0}\{x_\alpha\} \in \mathbf{L}$. ■

Finally, note that (which I have learned from A.B. Harazišvili), if, for $F : \mathbf{R}^2 \to \mathbf{R}$, the function $\psi_{\langle F,f,g\rangle} : \mathbf{R} \to \mathbf{R}$ given by $\psi_{\langle F,f,g\rangle}(x) = F(f(x), g(x))$ is (L)-measurable for any (L)-measurable $f, g : \mathbf{R} \to \mathbf{R}$, then F is (L)-measurable. This follows from the fact that there is a bijection $H : \mathbf{R}^2 \to \mathbf{R}$ such that both H and H^{-1} map (L)-measurable sets onto (L)-measurable sets (the analogous facts hold for (B)-measurability).

2. Comparing Measurability and Sup-Measurability for Measure and Category. For $F : \mathbf{R}^2 \to \mathbf{R}$, consider the following four properties:

(1) F is (L)-measurable,

(2) F is (B)-measurable,

(3) F is (L)-sup-measurable,

(4) F is (B)-sup-measurable.

Define a sequence $\langle i_1, i_2, i_3, i_4 \rangle \in \{0,1\}^4$ (depending on F) by

$$i_k = \begin{cases} 1 & \text{if } (k) \text{ is fulfilled,} \\ 0 & \text{otherwise} \end{cases} \qquad \text{for } k = 1,2,3,4.$$

We then say that F *is of type* $\langle i_1, i_2, i_3, i_4 \rangle$.

Our aim is to prove

2.1. Theorem. *For each $\langle i_1, i_2, i_3, i_4 \rangle \in \{0,1\}^4$, there exists an $F : \mathbf{R}^2 \to \{0,1\}$ of type $\langle i_1, i_2, i_3, i_4 \rangle$ (in some cases we assume that non $(\mathbf{K}) = c$ or non $(\mathbf{L}) = c$).*

We need the following

2.2. Lemma (cf. [O], 1.6). *There are an F_σ set $A \in \mathbf{K}$ and a G_δ set $B \in \mathbf{L}$, such that $A \cap B = \emptyset$ and $A \cup B = \mathbf{R}$. For any Bernstein set H in $\mathbf{R}$, the sets $C = H \cap A$ and $D = H \cap B$ are meager non-(L)-measurable and non-(B)-measurable of measure zero, respectively.* ■

Proof of Theorem 2.1. Fix Bernstein sets M and H in $\mathbf{R}^2$ and $\mathbf{R}$, respectively. Let A, B, C and D have the meanings as in Lemma 2.2.

The case of $\langle 1,1,1,1 \rangle$ is trivial (we put $F(x) = 1$ for all x). In the case of $\langle 0,0,0,0 \rangle$ we define F as the characteristic function of M. To show that F is as desired, it is enough to use Lemma 1.2 and observe that the set

$$\{x \in \mathbf{R} : F(x,0) = 1\} = \{x \in \mathbf{R} : \langle x, 0 \rangle \in M\}$$

forms a Bernstein set in $\mathbf{R}$, hence $g_{\langle F,f\rangle}$ for $f = 0$ is neither (L)-measurable nor (B)-measurable.

In the cases of $\langle 1,1,0,1\rangle$, $\langle 1,1,1,0\rangle$ and $\langle 1,1,0,0\rangle$ we define F as the characteristic functions of $C \times \{0\}$, $D \times \{0\}$ and $H \times \{0\}$, respectively, and we look at $g_{\langle F,f\rangle}$ for $f = 0$ by using Lemmas 1.2 and 2.2.

In the cases of $\langle 0,1,0,1\rangle$ and $\langle 1,0,1,0\rangle$ we define F as the characteristic functions of $M\cap(A\times\mathbf{R})$ and $M\cap(B\times\mathbf{R})$, respectively. If we add our functions associated with $\langle 0,1,0,1\rangle$ and $\langle 1,1,1,0\rangle$ (respectively, with $\langle 1,0,1,0\rangle$ and $\langle 1,1,0,1\rangle$), we get the function associated with $\langle 0,1,0,0\rangle$ (respectively, with $\langle 1,0,0,0\rangle$).

In the remaining cases we assume that non $(\mathbf{K}) = c$ or non $(\mathbf{L}) = c$. In the case of $\langle 1,0,1,1\rangle$ we assume that non $(\mathbf{K}) = c$ and define F as the function from 1.8 multiplied by the characteristic function of $B \times \mathbf{R}$. Having assumed that non $(\mathbf{K}) = c$, if we add the function associated with $\langle 1,1,0,1\rangle$ (resp. with $\langle 0,1,0,1\rangle$) to the F just defined, we obtain the function associated with $\langle 1,0,0,1\rangle$ (resp. with $\langle 0,0,0,1\rangle$).

In the case of $\langle 0,1,1,1\rangle$ we assume that non $(\mathbf{L}) = c$ and define F as the function from 1.8 multiplied by the characteristic function of $A \times \mathbf{R}$. Having assumed that non $(\mathbf{L}) = c$, if we add the function associated with $\langle 1,1,1,0\rangle$ (resp. with $\langle 1,0,1,0\rangle$) to the F just defined, we obtain the function associated with $\langle 0,1,1,0\rangle$ (resp. with $\langle 0,0,1,0\rangle$).

In the case of $\langle 0,0,1,1\rangle$ we assume that non $(\mathbf{K}) =$ non $(\mathbf{L}) = c$ and adding the functions associated with $\langle 1,0,1,1\rangle$ and $\langle 0,1,1,1\rangle$ we get the desired F.

By the disjointness of the respective sets, each addition considered above always yields a characteristic function. ■

3. Quasi-Sup-Measurability. For several cases in Theorem 2.1, the question arises whether the assumptions non $(\mathbf{K}) = c$ and non $(\mathbf{L}) = c$ can be eliminated (compare Problem 1.10). Taking this into account, one can try to replace (L)-sup-measurability and (B)-sup-measurability by weaker conditions in order to get the analogue of 2.1 without assuming non $(\mathbf{K}) = c$ and non $(\mathbf{L}) = c$. Now, we propose some conditions which are candidates to these roles. However we still cannot improve Theorem 2.1.

For fixed $F : \mathbf{R}^2 \to \mathbf{R}$, $f : \mathbf{R} \to \mathbf{R}$ and $y \in \mathbf{R}$ we define $\phi_{\langle F,f,y\rangle} : \mathbf{R} \to \mathbf{R}$ by $\phi_{\langle F,f,y\rangle}(x) = F(x, f(x) + y)$, $x \in \mathbf{R}$. We say that F is *quasi-(L)-sup-measurable*, if for each Borel $f : \mathbf{R} \to \mathbf{R}$ there is a countable $E \subseteq \mathbf{R}$ such that $\phi_{\langle F,f,y\rangle}$ is (L)-measurable for all $y \in \mathbf{R} \setminus E$. Obviously, each (L)-sup-measurable function is quasi-(L)-sup-measurable. As we shall prove, the converse is false. Similar remarks should be done for *quasi-(B)-sup-measurability* which we define analogously.

3.1. Proposition. *There exists an* $F : \mathbf{R}^2 \to \{0,1\}$ *which is*

(a) *(L)-measurable,*

(b) *(B)-measurable,*

(c) *(B)-sup-measurable,*

(d) *quasi-(L)-sup-measurable,*

(e) *non-(L)-sup-measurable.*

Proof. Define F as the characteristic function of $C \times \{0\}$ where C is taken from Lemma 2.2. It suffices to check (d) (compare the case of $\langle 1,1,0,1\rangle$ in the proof of 2.1). Consider any Borel $f : \mathbf{R} \to \mathbf{R}$. Observe that $E = \{y \in \mathbf{R} : f^{-1}[\{-y\}] \notin \mathsf{L}\}$ is countable since there is no uncountable disjoint family of Borel sets of positive measure. For each $y \notin E$, $\phi_{\langle F,f,y\rangle}$ is the characteristic function of $C \cap f^{-1}[\{-y\}]$ hence it equals 0 almost everywhere and, consequently, is (L)-measurable. ■

In an analogous way we prove

3.2. Proposition. *There exists an* $F : \mathbf{R}^2 \to \{0,1\}$ *which is*

(a) *(L)-measurable,*

(b) *(B)-measurable,*

(c) *(L)-sup-measurable,*

(d) *quasi-(B)-sup-measurable,*

(e) *non-(B)-sup-measurable.* ■

We say that a function $F : \mathbf{R}^2 \to \mathbf{R}$ is of *(q)-type* $\langle i_1, i_2, i_3, i_4\rangle \in \{0,1\}^4$ if

$$i_k = \begin{cases} 1 & \text{if } (k') \text{ is fulfilled,} \\ 0 & \text{otherwise} \qquad\qquad \text{for } k = 1,2,3,4 \end{cases}$$

where conditions (1′), (2′) are the same as (1), (2) and (3′) and (4′) state that F is quasi-(L)-sup-measurable and quasi-(B)-sup-measurable, respectively.

3.3. Theorem. *For each* $\langle i_1, i_2, i_3, i_4\rangle \in \{0,1\}^4$, *there exists an* $F : \mathbf{R}^2 \to \{0,1\}$ *of* (q)*-type* $\langle i_1, i_2, i_3, i_4\rangle$; *we assume non* $(\mathbf{K}) = c$ *and non* $(\mathbf{L}) = c$ *in the same cases where they were assumed in 2.1.*

Proof. Fix a Bernstein set H in $\mathbf{R}$ and let A, B, C, D have the meanings as in Lemma 2.2. Fix an uncountable $P \in \mathbf{K} \cap \mathbf{L}$.

In the case of $\langle 1,1,0,1\rangle$, $\langle 1,1,1,0\rangle$ and $\langle 1,1,0,0\rangle$ we define F as the characteristic functions of $C \times P$, $D \times P$ and $H \times P$, respectively, and look at $\phi_{\langle F,f,y\rangle}$ for $f = 0$ and $y \in P \setminus E$, where $E \subseteq \mathbf{R}$ is any countable set, by using Lemmas 1.2 and 2.2.

In the remaining cases we construct F in the same way as in the proof of 2.1. It is easy to verify that all additions used as for 2.1 lead to characteristic functions; this follows from the disjointness of the respective sets. ■

The definition of quasi-sup-measurability was inspired by the theorem of Pawlikowski who constructed in [P] a Borel set $U \subseteq \mathbf{R}^2$ such that all its vertical sections have measure zero and, for each Borel function $f : \mathbf{R} \to \mathbf{R}$ for all but countably many $y \in \mathbf{R}$ the set $\{x \in \mathbf{R} : \langle x, f(x) + y\rangle \in U\}$ is comeager. Our original proof of Proposition 3.2 was based on that fact but the present one is much simpler.

The notion of quasi-sup-measurability can further be generalized as follows (let us show this, for instance, in the case of quasi-(L)-sup-measurability). Let $\mathcal{T}$ be a σ-ideal of subsets of $\mathbf{R}$. We say that $F : \mathbf{R}^2 \to \mathbf{R}$ is $(\mathcal{T})$*-quasi-*(L)*-sup-measurable* if, for each Borel $f : \mathbf{R} \to \mathbf{R}$, there is $E \in \mathcal{T}$ such that $\phi_{\langle F,f,y\rangle}$ is (L)-measurable for all $y \in \mathbf{R} \setminus E$. In particular, in the cases $\mathcal{T} = \{\emptyset\}$ and $\mathcal{T} = \mathcal{T}_0$ we get the notions of (L)-sup-measurability and quasi-(L)-sup-measurability, respectively. Observe that, if quasi-(L)-sup-measurability and quasi-(B)-sup-measurability are replaced by $(\mathcal{T})$-quasi-(L)-sup-measurability and $(\mathcal{T})$-quasi-(B)-sup-measurability in 3.1 and 3.2, respectively, we get the true statements, provided that $\mathcal{T} \neq \{\emptyset\}$. Similarly, if $\mathbf{K} \cap \mathbf{L} \setminus \mathcal{T} \neq \emptyset$, the proof of 3.3 works for the respective version using $(\mathcal{T})$-quasi-sup-measurability instead of quasi-sup-measurability.

Finally, note that the $(\mathcal{T})$-quasi-(L)-sup-measurability of $F : \mathbf{R}^2 \to \mathbf{R}$ is equivalent to its measurability with respect to the σ-field of all $E \subseteq \mathbf{R}^2$ admitting $M \in \mathcal{T}$ such that $\{x \in \mathbf{R} : \langle x, f(x)+y\rangle \in E\}$ is (L)-measurable for all Borel $f : \mathbf{R} \to \mathbf{R}$ and all $y \in \mathbf{R} \setminus M$. The proof is immediate. A similar fact has already been observed in [S], Lemma 1, for an abstract case of sup-measurability. Since all functions F constructed in our paper take values 0 and 1 only, it is easy to formulate the results in the language of σ-fields.

Acknowledgements. I am grateful to A.B. Harazišvili, M. Penconek, A. Rosłanowski and W. Wilczyński for their helpful remarks.

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Received May 24, 1991