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ON d-MEASURE AND d-DIMENSION

1. Introduction

Let $\mathcal{G}$ be the graph of the Weierstrass function $W_\alpha(t) = \sum_{n=0}^{\infty} b^{-n\alpha} \cos b^n t$ with b an integer such that $b \geq 2$ and with $0 < \alpha < 1$. It is well-known that the Hausdorff dimension of $\mathcal{G}$, $HD(\mathcal{G})$, is equal or less than $2 - \alpha$ ([2]). But it has not been proved whether $HD(\mathcal{G})$ is exactly equal to $2 - \alpha$. In 1988, F. Rezakhanlou observed that packing dimension of $\mathcal{G}$, $PD(\mathcal{G})$, is $2 - \alpha$([7]). In [11], $HD(E) \leq \widehat{\underline{Cap}}(E) \leq \widehat{\overline{Cap}}(E) = PD(E)$ for $E \subset \mathbf{R^d}$ was shown, where $\widehat{\underline{Cap}}(\widehat{\overline{Cap}})$ is a dimensional index induced by lower (upper) capacity $\underline{Cap}$ ($\overline{Cap}$). Naturally we are interested in the value of $\widehat{\underline{Cap}}(\mathcal{G})$. Investigating $\widehat{\underline{Cap}}$, we find that unlike HD or PD, $\widehat{\underline{Cap}}$ has not been defined in terms of measure, and this can present difficulties in its theoretical development. Hence, in this paper, we define an outer measure, d-measure, whose dimensional index, d-dimension, is equal to $\widehat{\underline{Cap}}$. We examine its properties and examples, some of which are related to Hausdorff measure. On the other hand, we show $\widehat{\underline{Cap}}(\mathcal{G})$ is $2 - \alpha$, although d-measure does not attribute directly to find its value. Further we reprove that $PD(\mathcal{G})$ is $2 - \alpha$ by a different method from Rezakhanlou's (cf. [7]), and show that packing dimension of the Kiesswetter's curve is $\frac{3}{2}$ (cf. [1]). Finally we show that $\widehat{\underline{Cap}}(\mathcal{R}) = PD(\mathcal{R}) = 2 - \alpha$ for the graph $\mathcal{R}$ of the sum of Rademacher function (cf. [8]).

2. Preliminaries

For $\alpha > 0$, we define a pre-measure D^α of a bounded set F in $\mathbf{R^d}$ by $D^\alpha(F) = \liminf_{r \to 0} N(F, r) r^\alpha$, where $N(F, r)$ is the minimum number of closed balls in $\mathbf{R^d}$ with diameter r, needed to cover F. Then $D^\alpha(\phi) = 0$, $D^\alpha(F) = D^\alpha(\overline{F})$ for a bounded set F, and D^α is monotone.

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We employ Method I by Munroe to obtain an outer measure d^α of $E \subset \mathbf{R^d}$:

$$d^\alpha(E) = \inf\{\sum_{n=1}^{\infty} D^\alpha(E_n) : E \subset \cup_{n=1}^{\infty} E_n, E_n \text{ are bounded in } \mathbf{R^d}\}.$$

Henceforth, we call d^α the α–dimensional d–measure.

We define a rarefaction index δ of a bounded set F by

$$\begin{aligned}\delta(F) &= \sup\{\alpha > 0 : D^\alpha(F) = \infty\}\\ &= \inf\{\alpha > 0 : D^\alpha(F) = 0\}.\end{aligned}$$

Clearly δ is monotone, but not σ–stable in the sense of [11]. As in [11], we define a dimensional index induced by δ, $\hat{\delta}(E) = \inf\{\sup_n \delta(E_n) : E = \cup_{n=1}^{\infty} E_n, E_n$ are bounded $\}$ for any $E \subset \mathbf{R^d}$. Then $\hat{\delta}$ is clearly σ-stable. We recall the lower(upper) capacity $\underline{Cap}$ ($\overline{Cap}$) of a bounded set F,

$$\underline{Cap}(F) = \liminf_{r\to 0} \frac{\log N(F,r)}{-\log r}$$

$$(\,\overline{Cap}(F) = \limsup_{r\to 0} \frac{\log N(F,r)}{-\log r}\,) \text{ (cf. [6])}.$$

For the lower(upper) capacity $\underline{Cap}(\overline{Cap})$, we define $\widehat{\underline{Cap}}(\widehat{\overline{Cap}})$ as above. We also define a dimensional index, d-dimension of E, by

$$\begin{aligned}d - \dim(E) &= \sup\{\alpha > 0 : d^\alpha(E) = \infty\}\\ &= \inf\{\alpha > 0 : d^\alpha(E) = 0\}.\end{aligned}$$

We close this section by listing some notations.

$\mathcal{H}^\alpha(E)$; α-dimensional Hausdorff measure of E.

$p^\alpha(E)$; α-dimensional packing measure of E.

$|E|$; the diameter of E.

Γ; curve (i.e. the image of a continuous injection of a closed interval).

$\mathcal{L}(\Gamma)$; the length of a curve Γ.

$\mathcal{B}(\mathbf{R^d})$; the family of all bounded sets in $\mathbf{R^d}$.

3. Some properties of d–measure

We now start to prove the following basic properties of our new outer measure d^α.

Theorem 1. *d^α is Borel regular. That is, for any set $E \subset \mathbf{R}^d$, there exists a Borel set $B \subset \mathbf{R^d}$ such that $E \subset B$ and $d^\alpha(E) = d^\alpha(B)$.*

Proof. Since $N(F,r) = N(\overline{F},r), D^\alpha(F) = D^\alpha(\overline{F})$. Thus

$$d^\alpha(E) = \inf\{\sum_{n=1}^{\infty} D^\alpha(\overline{E}_n) : E \subset \cup_{n=1}^{\infty} E_n, E_n \in \mathcal{B}(\mathbf{R^d})\}.$$

For every integer $n > 0$, there exists a sequence of bounded sets $\{E_{n,i}\}_{i=1}^{\infty}$ such that $E \subset \cup_{i=1}^{\infty} E_{n,i}$ and $\sum_{i=1}^{\infty} D^\alpha(E_{n,i}) \leq d^\alpha(E) + \frac{1}{n}$. Let $B = \cap_{n=1}^{\infty} \cup_{i=1}^{\infty} \overline{E}_{n,i}$. Then $E \subset B$, and $B \subset \cup_{i=1}^{\infty} \overline{E}_{n,i}$ for every n. Therefore $d^\alpha(B) \leq \sum_{i=1}^{\infty} D^\alpha(\overline{E}_{n,i})$ for every n. Hence $d^\alpha(B) \leq \inf_n \sum_{i=1}^{\infty} D^\alpha(\overline{E}_{n,i}) = d^\alpha(E)$. The opposite inequality follows from monotonicity of d^α.

Theorem 2. *d^α is a metric outer measure.*

Proof. Plainly we have $d^\alpha(A \cup B) \leq d^\alpha(A) + d^\alpha(B)$ by subadditivity of d^α.

Suppose that dist $(E,F) > 0$ for $E, F \in \mathcal{B}(\mathbf{R^d})$. Then dist $(E,F) > 2\varepsilon > 0$ for some positive constant ε. Noting $N(E \cup F, \varepsilon) = N(E,\varepsilon) + N(F,\varepsilon)$, we have

$$D^\alpha(E \cup F) \geq D^\alpha(E) + D^\alpha(F).$$

Hence, for A and B with dist $(A,B) > 0$,

$$\begin{aligned} d^\alpha(A \cup B) &= \inf\{\sum D^\alpha(E_n) : A \cup B = \cup E_n, E_n \in \mathcal{B}(\mathbf{R^d})\} \\ &\geq \inf\{\sum D^\alpha(E_n \cap A) + \sum D^\alpha(E_n \cap B) : A \cup B = \cup E_n, \\ &\quad E_n \in \mathcal{B}(\mathbf{R^d})\} \\ &\geq \inf\{\sum D^\alpha(E_n \cap A) : A \cup B = \cup E_n, E_n \in \mathcal{B}(\mathbf{R^d})\} \\ &\quad + \inf\{\sum D^\alpha(E_n \cap B) : A \cup B = \cup E_n, E_n \in \mathcal{B}(\mathbf{R^d})\} \\ &\geq d^\alpha(A) + d^\alpha(B). \end{aligned}$$

Corollary 3. *d^α is a regular outer measure.*

Proof. It is immediate from Theorems 1 and 2 because the family of all measurable sets of a metric outer measure contains all Borel sets.

Theorem 4. *$\mathcal{H}^\alpha(E) \le d^\alpha(E)$ for every set E in $\mathbf{R^d}$.*

Proof. $\mathcal{H}^\alpha(F) \le D^\alpha(F)$ for any $F \in \mathcal{B}(\mathbf{R^d})$ from the definitions. By the subadditivity of $\mathcal{H}^\alpha$ and the prior fact, we easily obtain the result.

Corollary 5. *Suppose that for some α, there are numbers $c > 0$ and $\delta > 0$ such that $d^\alpha(E \cap U) \le c|U|^\alpha$ for all convex sets U with $|U| \le \delta$. Then $\mathcal{H}^\alpha(E) \ge d^\alpha(E)/c$. In particular, for $c = 1, \mathcal{H}^\alpha(E) = d^\alpha(E)$.*

Proof. It follows immediately from mass distribution principle 4.2 of [3]. When $c = 1$, it is obvious from Theorem 4.

Example 6. Let C be the well-known Cantor ternary set. Then we easily see that $d^\alpha(C \cap U) \le D^\alpha(C \cap U) \le |U|^\alpha$ for all intervals U, where $\alpha = \log 2/\log 3$. Hence $\mathcal{H}^\alpha(C) = d^\alpha(C)$ from Corollary 5. (Compare this with Example 10.)

Theorem 7. *$D^1(\Gamma) = \mathcal{L}(\Gamma)$ for a curve Γ.*

Proof. First, note that Theorem 4 and $d^1(\Gamma) \le D^1(\Gamma)$ give us $\mathcal{H}^1(\Gamma) \le D^1(\Gamma)$. Thus $\mathcal{L}(\Gamma) \le D^1(\Gamma)$ since $\mathcal{H}^1(\Gamma) = \mathcal{L}(\Gamma)$ (Lemma 3.2 of [2]).

Second, suppose that $\mathcal{L}(\Gamma) < \infty$. Then we can dissect Γ into n–parts with the same arc length: $\Gamma_1, \Gamma_2, \cdots$, and Γ_n. Let x_i be the midpoint of Γ_i for each $i = 1, 2, \cdots, n$. Then each Γ_i is contained in the closed ball of center x_i with radius $\mathcal{L}(\Gamma)/2n$. Thus $N(\Gamma, \mathcal{L}(\Gamma)/n)\mathcal{L}(\Gamma)/n \le n(\mathcal{L}(\Gamma)/n) = \mathcal{L}(\Gamma)$. Therefore $\liminf_{n\to\infty} N(\Gamma, \mathcal{L}(\Gamma)/n)\mathcal{L}(\Gamma)/n \le \mathcal{L}(\Gamma)$. Hence $D^1(\Gamma) \le \mathcal{L}(\Gamma)$.

Corollary 8. *$d^1(\Gamma) = \mathcal{L}(\Gamma)$ for a curve Γ.*

Remark 9. Noting that a curve Γ is strongly regular (cf. Theorem 6.1 of [10]), we have $\mathcal{L}(\Gamma) = \mathcal{H}^1(\Gamma) = d^1(\Gamma) = D^1(\Gamma) = p^1(\Gamma)$. There are many examples of E such that $0 < \mathcal{H}^\alpha(E) = d^\alpha(E) < \infty$ for some $\alpha > 0$, and also there are examples of E so that $\mathcal{H}^\alpha(E) < d^\alpha(E)$ for some E in R^d. (Corollary 14 in this paper, and [11]).

Example 10. Let C be as in Example 6. Then $\mathcal{H}^s(C) = d^s(C) = D^s(C) = 1$, where $s = \log 2/\log 3$. Clearly $N(C, 3^{-n})(3^{-n})^s = 2^n 2^{-n} = 1$, implying $D^s(C) \le 1$. Noting that $\mathcal{H}^s(C) = 1$ (Theorem 1.14 of [2]) and $\mathcal{H}^s(C) \le d^s(C)$ (Theorem 4), we obtain the result.

Example 11. Let E be the product of same Cantor–like sets ; $E = \{(x, y); x$

$= \sum_{n=1}^{\infty} \frac{a_n}{4^n}, y = \sum_{n=1}^{\infty} \frac{b_n}{4^n}$, where $a_n, b_n \in \{0, 3\}\}$. It was shown in [5] that $\mathcal{H}^1(E) = 2^{\frac{1}{2}}$. But $D^1(E) \le \liminf_{n\to\infty} N(E, 2^{\frac{1}{2}}\, 4^{-n}) 2^{\frac{1}{2}}\, 4^{-n} = 2^{\frac{1}{2}}$. Hence $\mathcal{H}^1(E) = d^1(E) = D^1(E) = 2^{\frac{1}{2}}$.

4. Capacity and d–dimension

In this section, we prove $\widehat{\underline{Cap}} = d - \dim$ by showing $\underline{Cap} = \delta$ (hence $\widehat{\underline{Cap}} = \hat{\delta}$) and $\hat{\delta} = d - \dim$.

Theorem 12.

$$\delta(F) = \underline{Cap}(F) \text{ for } F \in \mathcal{B}(\mathbf{R^d})$$

Proof. Suppose that $\underline{Cap}(F) > \delta(F) + \varepsilon$ for some $\varepsilon > 0$. Then there exists $\rho > 0$ such that for any $r \le \rho$, $\log N(F, r) > \log r^{-[\delta(F)+\varepsilon]}$; i.e. $N(F, r) r^{\delta(F)+\varepsilon} > 1$. Hence $D^{\delta(F)+\varepsilon}(F) \ge 1$, which is a contradiction. Therefore $\underline{Cap}(F) \le \delta(F) + \varepsilon$ for any $\varepsilon > 0$. Similarly we obtain $\underline{Cap}(F) \ge \delta(F) - \varepsilon$ for any $\varepsilon > 0$.

Theorem 13. $\hat{\delta}(E) = d - \dim(E)$ *for any set* E *in* $\mathbf{R^d}$.

Proof. Suppose that $\hat{\delta}(E) < d - \dim(E)$. Then there exists $\alpha \in (\hat{\delta}(E), d - \dim(E))$. So there is a sequence $\{E_n\}$ of bounded subsets of E such that $\cup E_n = E$ and $\sup_n \delta(E_n) < \alpha$. Thus $D^\alpha(E_n) = 0$ for any n, implying $d^\alpha(E) = 0$. It is a contradiction. Now, suppose $d - \dim(E) < \hat{\delta}(E)$. Then there is α such that $d - \dim(E) < \alpha < \hat{\delta}(E)$. Thus $d^\alpha(E) = 0$. Therefore, there exists a sequence $\{E_n\}$ of bounded subsets of E such that $E = \cup_{n=1}^{\infty} E_n$ and $D^\alpha(E_n) < \infty$ for every n, hence $\delta(E_n) \le \alpha$ for every n. Thus $\hat{\delta}(E) \le \alpha$. It is also a contradiction.

Corollary 14. $\widehat{\underline{Cap}}(E) = d - \dim(E)$ *for any* $E \subset \mathbf{R^d}$.

Remark 15. (a) In Example 10, we have seen that $\mathcal{H}^s(C) = d^s(C) = D^s(C) = 1$, where $s = \log 2/\log 3$. Hence, by Theorem 12 and Corollary 14, $\underline{Cap}(C) = \widehat{\underline{Cap}} = \log 2/\log 3$. But there is another way to show $\underline{Cap} = \widehat{\underline{Cap}} = s$ without relying on the fact $\mathcal{H}^s(C) = 1$: $d^s(C) \le D^s(C) = 1$ and $d^t(C) = \infty$ for $t < s = \log 2/\log 3$, which can be seen easily from the Baire Category theorem.

(b) Let $f : F \to \mathbf{R}^m$ satisfy a Hölder condition $|f(x) - f(y)| \le c|x - y|^\alpha (x, y \in F)$. Then $d^{\frac{s}{\alpha}}(f(F)) \le c^{\frac{s}{\alpha}} d^s(F)$ because $D^{\frac{s}{\alpha}}(f(E)) \le c^{\frac{s}{\alpha}} D^s(E)$ for any bounded subset E of F. Hence $\widehat{\underline{Cap}}(= d - \dim)$ is invariant under bi-Lipschitz transformations. Besides, one could show many other properties of

$\underline{\widehat{Cap}}$ using d-measure.

On the other hand, we can obtain d-dim or PD of some sets using capacity.

Theorem 16. *Let $f : [0,1] \to \mathbf{R}$, and let G be its graph. Suppose that $0 < \alpha < 1$. If there exist positive constants C_1 and C_2 such that $C_1|J|^\alpha \leq \sup_{x,y\in J}|f(x) - f(y)| \leq C_2|J|^\alpha$ for any interval J in $[0,1]$, then $d - \dim(G) = PD(G) = 2 - \alpha$.*

Proof. By Remark 9 of [6], $\underline{Cap}(G) = \overline{Cap}(G) = 2 - \alpha$. If $G = \cup_{n=1}^\infty E_n$, then $\cup_{n=1}^\infty E_n = \cup_{n=1}^\infty \overline{E}_n$ since G is a closed set in $\mathbf{R}^2$. Then, by the Baire Category theorem, there exists $\overline{E}_m \subset G$ such that $\mathrm{Int}_G(\overline{E}_m) \neq \emptyset$, where $\mathrm{Int}_G(\overline{E}_m)$ is the interior of $\overline{E}_m$ in G. Thus there exists $r > 0$ such that $B_r(z) \cap G \subset \mathrm{Int}_G(\overline{E}_m)$ for a fixed $z \in \mathrm{Int}_G(\overline{E}_m)$, where $B_r(z)$ is the open disc in $\mathbf{R}^2$ with center z and radius r. But there exists an interval $J \subset [0,1]$ with very small length such that $\mathrm{Proj}_{\text{X-axis}}(z) \in J$, and $\sup_{x,y\in J}|f(x) - f(y)| \leq C_2|J|^\alpha < \frac{r}{4}$. Thus $\mathrm{Int}_G(\overline{E}_m) \supset \{(a, f(a)) : a \in J\}$. Hence $\underline{Cap}(\mathrm{Int}_G(\overline{E}_m)) \geq 2 - \alpha$. Therefore $\underline{\widehat{Cap}}(G) \geq 2 - \alpha$. By Corollary 14, $d - \dim(G) = \underline{\widehat{Cap}}(G) \geq 2 - \alpha$. Whence $2 - \alpha \leq d - \dim(G) \leq PD(G) = \overline{\widehat{Cap}}(G) \leq \overline{Cap}(G) \leq 2 - \alpha$ [10].

Remark 17. Let f be a continuous function defined on $[0,1]$ and let G_J be the graph of f over $J \subset [0,1]$. If f satisfies the property that $\underline{Cap}(G_J) = \beta$ ($\overline{Cap}(G_J) = \beta$) for any interval $J \subset [0,1]$ and a fixed $1 < \beta < 2$, then we easily see that $d - \dim(G) = \beta$ ($PD(G) = \beta$) for $G = G_{[0,1]}$ using the Baire Category theorem (c.f., Corollary 11.2 of [3] as an example).

Example 18. Let $\mathcal{G}$ be the graph of the Weirstrass function $W_\alpha(t)$. Then $PD(\mathcal{G}) = 2 - \alpha$ since $W_\alpha(t)$ satisfies the assumption of Theorem 16 (cf. [4], [6]). Furthermore, $d - \dim(\mathcal{G}) = 2 - \alpha$ in the same manner.

Example 19. Let $\mathcal{K}$ be the Kiesswetter's curve ([1]). Then $d - \dim(\mathcal{K}) = PD(\mathcal{K}) = \frac{3}{2}$. This is because the Kiesswetter function κ satisfies the assumptions of Theorem 16 ; $|J|^{\frac{1}{2}} \leq \sup_{x,y\in J}|\kappa(x) - \kappa(y)| \leq 3|J|^{\frac{1}{2}}$ for any interval $J \subset [0,1]$. (cf. $HD(\mathcal{K}) = \frac{3}{2}$ in [1])

Example 20. Let $0 < \alpha < 1$ and $f_\alpha(x) = \sum 2^{-\alpha i}R_i(x)$, where R_i is the i-th Rademacher function and $0 \leq x < 1$ ([6], [8]). Let G be its graph. Then it follows essentially from Proposition 2[6] and Theorem 16 that $d - \dim(G) = PD(G) = 2 - \alpha$. We note that $HD(G) < d - \dim(G) = PD(G) = 2 - \alpha$ for PV number 2^α ([6]).

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