

ON m -RINGS OF FUNCTIONS AND SOME GENERALIZATIONS OF THE NOTION OF DENSITY POINT

Abstract. In this paper we investigate properties of some rings and ideals of real functions. Moreover we present some generalizations of the notion of density point.

In 1985, there appeared a paper ([3]) by W. Poreda, E. Wagner-Bojakowska and W. Wilczyński in which a certain kind of density points was defined topologically. This paper presented a new method of showing parallels between the σ -ideals of first category sets and sets of Lebesgue measure zero, i.e., was a successive study of σ -ideals of sets. Further explorations of these ideas can be found in many interesting papers ([1],[4],[5],[6],[7]).

Along with ideals of sets, other small systems of sets were studied. In connection with this, small systems of functions were investigated. The first place we encountered this notion was in a paper from 1972 by Prof. B. Riečan [8]. Studying the results of investigations connected with these problems, one can observe that, in many places, what is essential is the algebraic structure of the classes of the transformations which are examined. Hence the themes of the present article concentrate around an algebraic approach to the problems of measurability, density points and other questions connected with these problems. Such an approach to the subject creates, on the one hand, new possibilities of discussing these problems in more abstract spaces; on the other hand, a close connection with the problems considered by, among others, the groups of Profs. W. Wilczyński, B. Riečan or L. Zajíček makes the facts presented here constitute, indeed, generalizations of the earlier results.

Throughout the paper, we use the classical symbols and notations. By the letters R and N we denote the real line and the set of all positive integers, resp. The symbol χ_A stands for a characteristic function of the set A .

For an arbitrary function f defined on $A \subset X$ we adopt

$$f^*(x) = \begin{cases} f(x) & \text{for } x \in A, \\ 0 & \text{for } x \notin A. \end{cases}$$

Given two sets A and B and $\alpha \in B$, the function $\text{const}_\alpha^{A,B}$ is the constant function from A into B with value α . If we omit A and B in this notation, then we assume that these sets are fixed beforehand.

We adopt $Z(f) = \{x : f(x) = 0\}$ and for arbitrary $\{f_n\}$ we denote by $\ell(f_n) = \{x : \lim_{n \rightarrow \infty} f_n(x) \text{ does not exist}\}$ (if, for example, $\lim_{n \rightarrow \infty} f_n(x) = +\infty$ we understand that this limit does not exist either).

We adopt $A \Delta B = (A \setminus B) \cup (B \setminus A)$.

Moreover, assume the following definitions:

We say that a sequence of functions $\{f_k\}$ is cofinal with a sequence of sets of functions $\{A_n\}$ if for every $n \in N$ there exists $k \in N$ such that $f_k \in A_n$.

We say that a sequence of functions $\{f_k\}$ is *-cofinal with a sequence of sets of functions $\{A_n\}$ if for every subsequence $\{f_{t_k}\}$ of $\{f_k\}$ there exists a subsequence $\{f_{s_{t_k}}\}$ cofinal with $\{A_n\}$ and $\chi_{\ell(f_{s_{t_k}})} \in A_n$ for every $n = 1, 2, \dots$.

To begin with, we shall deal with special kinds of rings of functions and ideals (in the algebraic sense) of these rings, and show a close connection between these objects and the ideas of σ -algebras of sets and set theoretic σ -ideals (in the sense of set theory).

DEFINITION. Let X be an arbitrary set. We say that a ring S with the unit, of real functions defined on X is an m -ring if:

1. $|f| \in S$ and $(\frac{1}{f})^* \in S$ for every $f \in S$;
2. if $f = \lim_{n \rightarrow \infty} f_n$, where $\{f_n\} \subset S$, then $f \in S$.

THEOREM 1. (a) Any m -ring S forms the family of all measurable functions with respect to the σ -algebra $\{A \subset X : \chi_A \in S\}$.

(b) Any family of all measurable functions with respect to some σ -algebra is m -ring.

Proof. (a) First we shall show that the family $\mathcal{A} = \{A \subset X : \chi_A \in S\}$ is σ -algebra of sets.

Assume that $A \in \mathcal{A}$. This means that $\chi_A \in S$ and so $-(\chi_A - \text{const}_1) \in S$. On the other hand $\chi_{X \setminus A} = -(\chi_A - \text{const}_1)$ which means that $X \setminus A \in \mathcal{A}$.

Now, let $\{A_n\} \subset \mathcal{A}$. We shall show that $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$, i.e.

$$(1) \quad \chi_{\bigcup_{n=1}^{\infty} A_n} \in S.$$

Notice that:

$$\chi_{P \cup Q} \in S, \text{ for every } P, Q \in \mathcal{A}.$$

Therefore, it is not hard to check that:

$$(2) \quad \chi_{\bigcup_{n=1}^k A_n} \in S \text{ for every positive integer } k.$$

Notice that:

$$\lim_{k \rightarrow \infty} \chi_{\bigcup_{n=1}^k A_n} = \chi_{\bigcup_{n=1}^{\infty} A_n}.$$

From the above, by (2) and condition 2 of the definition of an m -ring, we may infer that relationship (1) does take place.

We shall now show that S includes all measurable functions with respect to \mathcal{A} . Let us first observe that $\text{const}_q \in S$ for every rational number q . Therefore, by condition 2 of the definition of an m -ring, each constant function belongs to S . So, by the definition of \mathcal{A} , every simple measurable function belongs to S . Therefore, in view of 2 of the definition of m -ring, every measurable function with respect to \mathcal{A} belongs to S .

To finish the proof (a), it suffices to demonstrate that every function $f \in S$ is measurable with respect to \mathcal{A} . Let α be an arbitrary real number. Note that:

$$\chi_{\{x: f(x) > \alpha\}} = \left(\frac{1}{\max(f(x) - \text{const}_\alpha(x), 0)} \right)^* \max(f(x) - \text{const}_\alpha(x), 0).$$

Thus, by the condition 1 of the definition of m -ring, the function on the right side of the above equality belongs to S , and so $\{x : f(x) > \alpha\} \in \mathcal{A}$ which implies that f is measurable.

Proof (b) is immediate.

REMARK. By $\mathcal{A}(S)$ we denote the σ -algebra from (a) of the above Theorem and by $S(\mathcal{A})$ the m -ring of all functions measurable with respect to the fixed σ -algebra \mathcal{A} ((b) of Theorem 1).

DEFINITION. Let X be an arbitrary set and let S be an m -ring of real functions defined on X . We say that an ideal \mathcal{J} of S is an m -ideal, if the following conditions are fulfilled:

1. If $f = \lim_{n \rightarrow \infty} f_n$, where $\{f_n\}_{n=1}^{\infty} \subset \mathcal{J}$, then $f \in \mathcal{J}$.
2. If $|g| \leq |f|$, where $f \in \mathcal{J}$, then $g \in \mathcal{J}$.

THEOREM 2. (a) For every m -ideal \mathcal{J} of the m -ring S the family $J_{\mathcal{J}} = \{A \subset X : \chi_A \in \mathcal{J}\}$ is a σ -ideal of the σ -algebra $\mathcal{A}(S)$ (we say that $J_{\mathcal{J}}$ is the σ -ideal generated by \mathcal{J}).

(b) For every σ -ideal J of sets belonging to the σ -algebra \mathcal{A} the family $\mathcal{J}_J = \{f : X \rightarrow R : X \setminus Z(f) \in J\}$ is an m -ideal of the m -ring $S(\mathcal{A})$ (we say that \mathcal{J}_J is the m -ideal generated by J).

Proof. (a) Let $B \in J_{\mathcal{J}}$ and $A \subset B$. Then $\chi_B \in \mathcal{J}$. Moreover $|\chi_A| \leq |\chi_B|$, thus $\chi_A \in S$. Therefore $\chi_A = \chi_A \cdot \chi_B \in \mathcal{J}$.

Now let $\{A_n\} \subset J_{\mathcal{J}}$. We shall show that $\bigcup_{n=1}^{\infty} A_n \in J_{\mathcal{J}}$, i.e. $\chi_{\bigcup_{n=1}^{\infty} A_n} \in \mathcal{J}$. The method used to prove the above fact is analogous to the proof of Theorem 1(a). Because $J_{\mathcal{J}} \subset \mathcal{A}(S)$ the proof of (a) is finished.

(b) Remark that $\mathcal{J}_J \subset S(\mathcal{A})$. Indeed, let $f \in \mathcal{J}_J$ and let $\alpha \in R$. Consider $\{x : f(x) < \alpha\}$. We may consider two cases:

1°. $\alpha \leq 0$. Then $\{x : f(x) < \alpha\} \in J \subset \mathcal{A}$,

2°. $\alpha > 0$. Then $\{x : f(x) \geq \alpha\} \in J \subset \mathcal{A}$ and so $\{x : f(x) < \alpha\} = X \setminus \{x : f(x) \geq \alpha\} \in \mathcal{A}$.

From the above we may deduce that $f \in S(\mathcal{A})$.

Now, we shall show that \mathcal{J}_J is an m -ideal of $S(\mathcal{A})$. It is easy to see that \mathcal{J}_J is closed with respect to the addition of functions. So let $f \in S(\mathcal{A})$ and $g \in \mathcal{J}_J$. Then $Z(f \cdot g) \supset Z(g)$ and consequently $X \setminus Z(f \cdot g) \in J$. Therefore \mathcal{J}_J is an ideal of $S(\mathcal{A})$.

Now let $\{f_n\} \in \mathcal{J}_J$ and $f = \lim_{n \rightarrow \infty} f_n$. Then $Z(f) \supset \bigcap_{n=1}^{\infty} Z(f_n)$ and so $X \setminus Z(f) \in J$. Hence $f \in \mathcal{J}_J$.

To finish this proof it suffices to demonstrate that condition 2 of the definition of an m -ideal takes place. Let $f \in \mathcal{J}$ and $g : X \rightarrow R$ be a function such that $|g| \leq |f|$. Thus $Z(g) \supset Z(f)$ which means that $X \setminus Z(g) \in J$ and consequently $g \in \mathcal{J}_J \subset S(\mathcal{A})$, which finally completes the proof.

Since the algebraic structure is evident within the classes of functions we have considered, it is difficult to avoid asking about the algebraic structure of the m -ideals. For example, it is not hard to see that any m -ideal \mathcal{J} of any m -ring S is a linear space (if the product $\alpha \cdot f$ – where $\alpha \in R$ and $f \in \mathcal{J}$ – is interpreted as $\text{const}_{\alpha} \cdot f$). Consequently, it is interesting to ask about the dimension of this space. The answer to this question is included in the following theorem.

THEOREM 3. *Let \mathcal{J} be an arbitrary m -ideal of some m -ring S . Then the following conditions are equivalent:*

(i) $\dim \mathcal{J} = \infty$,

(ii) \mathcal{J} contains some function which assumes infinitely many values,

(iii) for σ -ideal $J_{\mathcal{J}}$ generated by \mathcal{J} ; $\text{card } J_{\mathcal{J}} \geq \aleph_0$.

Proof. (i) \Rightarrow (ii). Let x_1 be a point such that, there exists some function $f \in \mathcal{J}$, that $f(x_1) \neq 0$. Hence $\chi_{\{x_1\}} \in \mathcal{J}$.

Assume that we have pairwise distinct points x_1, \dots, x_{n-1} such that $i \cdot \chi_{\{x_i\}} \in \mathcal{J}$. Then the collection $\{i \cdot \chi_{\{x_i\}} : i = 1, \dots, n-1\}$ is linearly independent. Since $\dim \mathcal{J} = \infty$, then there exists $h \in \mathcal{J}$ such that $h, \chi_{\{x_1\}}, 2 \cdot \chi_{\{x_2\}}, \dots, (n-1) \cdot \chi_{\{x_{n-1}\}}$ is linearly independent. This means that there exists $x_n \notin \{x_1, \dots, x_{n-1}\}$ and $h(x_n) \neq 0$. It is not hard to check that $n \cdot \chi_{\{x_n\}} \in \mathcal{J}$.

Continuing this procedure we obtain infinite sequence $\{n \cdot \chi_{\{x_n\}}\} \in \mathcal{J}$ such that $x_i \neq x_j$ for $i \neq j$. Hence

$$\sum_{i=1}^n i \cdot \chi_{\{x_i\}} \in \mathcal{J} \text{ for every } n = 1, 2, \dots$$

This means that $\lim_{n \rightarrow \infty} \sum_{i=1}^n i \cdot \chi_{\{x_i\}} \in \mathcal{J}$, but

$$f = \lim_{n \rightarrow \infty} \sum_{i=1}^n i \cdot \chi_{\{x_i\}} = \begin{cases} 1 & \text{for } x = x_i \text{ (} i = 1, 2, \dots \text{)}, \\ 0 & \text{for } x \notin \{x_1, x_2, \dots\}, \end{cases}$$

and consequently, \mathcal{J} fulfills (ii).

(ii) \Rightarrow (iii). Let $f \in \mathcal{J}$ assume infinitely many values. Then $A = \{x : f(x) \neq 0\}$ is an infinite set and we may remark that

$$\chi_{\{x\}} \in \mathcal{J} \text{ for every } x \in A,$$

which means that $\{x\} \in J_{\mathcal{J}}$, for every $x \in A$.

(iii) \Rightarrow (i). From the assumption we have made, it follows that there exists an infinite set A such that $A \in J_{\mathcal{J}}$. Let a_1, a_2, \dots be a sequence of pairwise distinct point of A . Thus $\chi_{\{a_1\}}, \chi_{\{a_2\}}, \dots$ is a sequence of functions belonging to \mathcal{J} and this collection is linearly independent.

With many considerations carried out in the sequel, it is more advantageous to apply, instead of m -ideal of functions, some sequence of sets of functions whose intersection will be the given m -ideal. This situation resembles a connection of σ -ideals of sets and small systems of sets ([2]).

DEFINITION. Let S be an arbitrary m -ring of functions. A family $\mathcal{F} = \{F_n : n = 1, 2, \dots\}$ of subsets of S is called an m -system of the m -ring S , if $\{F_n\}_{n=1}^{\infty}$ is a decreasing sequence satisfying the following conditions:

- (i) If $f_i \in F_i$ ($i = n, n + 1, \dots, n + r$), then $\sum_{i=n}^{n+r} f_i \in F_{n-1}$.
- (ii) If $\{f_k\}_{k=1}^{\infty}$ is cofinal with $\{F_n\}_{n=1}^{\infty}$, then $g \cdot (\lim_{k \rightarrow \infty} f_k)^* \in F_n$, for every function $g \in S$ and for every $n \in N$.
- (iii) If for a function g there exists a sequence $\{f_k\}$ cofinal with $\{F_n\}$ such that $|g| \leq |f_k|$ (for $k = 1, 2, \dots$) and $\ell(f_k) \subset Z(g)$, then $g \in S$.
- (iv) If $f \in F_n$, $g \in S$, and $|g| \leq |f|$, then $g \in F_n$ (for $n = 1, 2, \dots$).

The idea of the above definition is connected with the definition of a small system of functions ([8]).

THEOREM 4. *Let $\{F_n : n = 1, 2, \dots\}$ be an m -system of the m -ring S . Then $\mathcal{J} = \mathcal{J}(\{F_n : n = 1, 2, \dots\}) = \bigcap_{n=1}^{\infty} F_n$ is an m -ideal of the m -ring S (we say that $\{F_n : n = 1, 2, \dots\}$ generates $\mathcal{J}(\{F_n : n = 1, 2, \dots\})$).*

Proof. Let $f, g \in \mathcal{J}$. Then $f, g \in F_n$ (for $n = 1, 2, \dots$) and according to (i), $f + g \in F_n$ (for $n = 1, 2, \dots$) which means that \mathcal{J} is closed with respect to the addition of functions.

Now let $f \in \mathcal{J}$ and $g \in S$. Condition (ii) implies that $g \cdot f \in \mathcal{J}$ and so \mathcal{J} is an ideal of S . Since conditions 1 and 2 of the definition of m -ideal immediately follow from (ii) and (iii), then this proof is finished.

It is easy to see that if \mathcal{J} is a fixed m -ideal of functions of some m -ring S , then, putting $F_n = \mathcal{J}$ (for $n = 1, 2, \dots$), we obtain an m -system $\{F_n : n = 1, 2, \dots\}$ such that $\bigcap_{n=1}^{\infty} F_n = \mathcal{J}$. Such an m -system is called a trivial m -system generating \mathcal{J} . If S is an m -ring and $\mathcal{J} = S$ is an m -ideal of the m -ring S , then the only m -system generating \mathcal{J} is, of course, a trivial m -system. However, it turns out that, for an m -ideal $\mathcal{J} \neq S$, there always exist non-trivial m -systems generating this m -ideal.

LEMMA 1. *Let f be an arbitrary function belonging to the m -ideal \mathcal{J} of the m -ring S and g be a function such that $\{x : g(x) \neq f(x)\} \in J_{\mathcal{J}}$. Then $g \in \mathcal{J}$.*

Proof. Infer that $g \in S$. Let $T = \{x : f(x) \neq g(x) \wedge f(x) = 0\}$. Of course $T \in J_{\mathcal{J}}$. Thus $\chi_T \in \mathcal{J}$ and $f + \chi_T \in \mathcal{J}$ and moreover $(f + \chi_T)(x) \neq 0$ for every x such that $f(x) \neq g(x)$. Consequently

$$g = g \left(\frac{1}{f + \chi_T} \right)^* (f + \chi_T) \in \mathcal{J}.$$

THEOREM 5. Let \mathcal{J} be an m -ideal of the m -ring S such that $\mathcal{J} \neq S$. Then there exists a non-trivial m -system $\{F_n : n = 1, 2, \dots\}$ of the m -ring S generating \mathcal{J} .

Proof. Let X be a domain of functions from S . Put:

$$F_n = \{f \in S : \exists g \in \mathcal{J} \sup_{x \in X} |f(x) - g(x)| \leq \frac{1}{2^n}\} \text{ for } n = 1, 2, \dots$$

We shall first prove that $\{F_n : n = 1, 2, \dots\}$ is an m -system. One can see that $\{F_n\}_{n=1}^\infty$ is a decreasing sequence of subsets of S . We shall now prove the veracity of (i) of the definition of m -system.

Let $f_i \in F_i$ ($i = n, n+1, \dots, n+r$). This means that there exist $g_i \in \mathcal{J}$ ($i = n, n+1, \dots, n+r$) such that

$$\sup_{x \in X} |f_i(x) - g_i(x)| \leq \frac{1}{2^i} \quad (i = n, n+1, \dots, n+r).$$

Hence $\sum_{i=n}^{n+r} g_i \in \mathcal{J}$ and moreover

$$\sup_{x \in X} \left| \sum_{i=n}^{n+r} f_i(x) - \sum_{i=n}^{n+r} g_i(x) \right| \leq \frac{1}{2^{n-1}},$$

so $\sum_{i=n}^{n+r} f_i \in F_{n-1}$.

To verify (ii) from the definition of an m -system, assume that $\{f_k\}_{k=1}^\infty$ is cofinal with $\{F_n\}_{n=1}^\infty$. We shall now prove that

$$(1) \quad \left(\lim_{k \rightarrow \infty} f_k\right)^* \in \mathcal{J}.$$

Let $g_{k_n} \in \mathcal{J}$ ($n = 1, 2, \dots$) be a function such that

$$\sup_{x \in X} |f_{k_n}(x) - g_{k_n}(x)| \leq \frac{1}{2^n}.$$

It is evident that

$$(2) \quad \ell(f_{k_n}) = \ell(g_{k_n})$$

and

$$\lim_{n \rightarrow \infty} f_{k_n}(x) = \lim_{n \rightarrow \infty} g_{k_n}(x) \text{ for } x \notin \ell(f_{k_n}) = \ell(g_{k_n}).$$

Put (for $n = 1, 2, \dots$)

$$h_{k_n}(x) = \begin{cases} g_{k_n}(x) & \text{if } x \notin \ell(f_{k_n}) = \ell(g_{k_n}), \\ 0 & \text{if } x \in \ell(f_{k_n}) = \ell(g_{k_n}). \end{cases}$$

Since $|h_{k_n}(x)| \leq |g_{k_n}(x)|$ ($n = 1, 2, \dots$) then $h_{k_n} \in S$ and so $h_{k_n} = h_{k_n} \cdot \left(\frac{1}{g_{k_n}}\right)^* \cdot g_{k_n} \in \mathcal{J}$ ($n = 1, 2, \dots$) and the following limit exists

$$\lim_{n \rightarrow \infty} h_{k_n} = \left(\lim_{n \rightarrow \infty} g_{k_n}\right)^*.$$

By (2),

$$\left(\lim_{n \rightarrow \infty} f_{k_n}\right)^* = \left(\lim_{n \rightarrow \infty} g_{k_n}\right)^* = \lim_{n \rightarrow \infty} h_{k_n} \in \mathcal{J}.$$

Infer that

$$\left|\left(\lim_{k \rightarrow \infty} f_k\right)^*\right| \leq \left|\left(\lim_{n \rightarrow \infty} f_{k_n}\right)^*\right|,$$

which means that (1) is satisfied.

By virtue of (1), we may infer that

$$g \cdot \left(\lim_{k \rightarrow \infty} f_k\right)^* \in \mathcal{J} \subset F_n \quad (n = 1, 2, \dots) \text{ for every } g \in S.$$

Assume that the assumptions of (iii) are true. We shall show that

$$(3) \quad |g| \leq \left|\left(\lim_{k \rightarrow \infty} f_k\right)^*\right|.$$

Indeed, if $x \in Z(g)$, then the inequality $|g(x)| \leq \left|\left(\lim_{k \rightarrow \infty} f_k\right)^*(x)\right|$ is evident. Suppose that $x \notin Z(g)$. Then there exists $\lim_{k \rightarrow \infty} f_k(x)$ and by the assumption: $|g(x)| \leq |f_k(x)|$ for $k = 1, 2, \dots$, we obtain $|g(x)| \leq \left|\lim_{k \rightarrow \infty} f_k(x)\right|$. This ends the proof of (3).

From the above reasoning $\left(\lim_{k \rightarrow \infty} f_k\right)^* \in \mathcal{J}$ and so $\left|\left(\lim_{k \rightarrow \infty} f_k\right)^*\right| = \left|\left(\lim_{k \rightarrow \infty} f_k\right)^*\right| \cdot \left(\frac{1}{\left(\lim_{k \rightarrow \infty} f_k\right)^*}\right)^* \cdot \left(\lim_{k \rightarrow \infty} f_k\right)^* \in \mathcal{J}$, which together with (3), and the fact that \mathcal{J} is an m -ideal of the m -ring S gives $g \in S$.

We shall now show that the condition (iv) from the definition of an m -system holds.

Let $n \in N$, $f \in F_n$, $g \in S$ and $|g| \leq |f|$. Then there exists $h \in \mathcal{J}$ such that $\sup_{x \in X} |f(x) - h(x)| \leq \frac{1}{2^n}$. Put

$$h'(x) = \begin{cases} h(x) & \text{for } x \notin Z(g), \\ 0 & \text{for } x \in Z(g). \end{cases}$$

Then $|h'| \leq |h|$ and so $h' \in S$. Moreover $h' = h' \cdot \left(\frac{1}{h}\right)^* \cdot h \in \mathcal{J}$. Put

$$h_1(x) = \begin{cases} g(x) & \text{for } x \notin Z(h'), \\ 0 & \text{for } x \in Z(h'). \end{cases}$$

Hence, according to Lemma 1, $h_1 \in \mathcal{J}$. It is not difficult to notice that $\sup_{x \in X} |g(x) - h_1(x)| \leq \frac{1}{2^n}$, which means that $g \in F_n$.

Infer now that $\{F_n : n = 1, 2, \dots\}$ generates \mathcal{J} . Of course $J \subset \bigcap_{n=1}^{\infty} F_n$. So let $f \in \bigcap_{n=1}^{\infty} F_n$. This means that for every $n \in N$ there exists $g_n \in \mathcal{J}$ such that $\sup_{x \in X} |f(x) - g_n(x)| \leq \frac{1}{2^n}$. Consequently $\lim_{n \rightarrow \infty} g_n = f$ and $f \in \mathcal{J}$.

We have presented a proof of the fact that $\{F_n : n = 1, 2, \dots\}$ is an m -system generating \mathcal{J} . We shall show that this m -system is non-trivial.

Let $h \notin \mathcal{J}$ be a bounded function belonging to S . Thus $|h(x)| \leq M$ for every $x \in X$. Consider the following function:

$$f_n = \frac{1}{2^n M} \cdot h \text{ for every } n \in N.$$

Then $|f_n(x)| \leq \frac{1}{2^n}$ for every $x \in X$, $n \in N$ and so $f_n \in F_n \setminus \mathcal{J}$ for every $n \in N$. This ends the proof of Theorem 5.

DEFINITION. Let S be some m -ring of functions and let \mathcal{J} be an m -ideal of S . We say that a sequence $\{f_k\} \subset S$, \mathcal{J} -tends to a function $f \in S$ (we write $\mathcal{J} - \lim_{k \rightarrow \infty} f_k = f$), if

$$(\lim_{k \rightarrow \infty} f_k)^* - f \in \mathcal{J}.$$

THEOREM 6. Let S be some m -ring of functions and let \mathcal{F} be an m -system of S and $\mathcal{J} = \mathcal{J}(\mathcal{F})$, then

- (i) If $\{f_k - f\}_{k=1}^{\infty}$ is cofinal with \mathcal{F} and $\chi_{\ell(f_k) \setminus Z(f)} \in \mathcal{J}$, then $\mathcal{J} - \lim_{k \rightarrow \infty} f_k = f$.
- (ii) If $\mathcal{J} - \lim_{k \rightarrow \infty} f_k = f$, then $(\lim_{k \rightarrow \infty} (f_k - f))^* \in \mathcal{J}$.

Proof. (i) From the assumptions we may infer that $(\lim_{k \rightarrow \infty} (f_k - f))^* \in \mathcal{J}$. Since $\chi_{\ell(f_k) \setminus Z(f)} \in \mathcal{J}$, then

$$\{x : (\lim_{k \rightarrow \infty} (f_k - f))^*(x) \neq ((\lim_{k \rightarrow \infty} f_k)^* - f)(x)\} \in J_{\mathcal{J}},$$

which, according to Lemma 1, means that $(\lim_{k \rightarrow \infty} f_k)^* - f \in \mathcal{J}$.

(ii) First we shall remark that

$$(*) \quad ((\lim_{k \rightarrow \infty} f_k)^* - f)(x) \neq 0, \text{ for every } x \in \ell(f_k) \setminus Z(f).$$

Put now

$$h(x) = \begin{cases} ((\lim_{k \rightarrow \infty} f_k)^* - f)(x) & \text{for } x \in \ell(f_k) \setminus Z(f), \\ 0 & \text{for } x \notin \ell(f_k) \setminus Z(f). \end{cases}$$

Then $h \in \mathcal{J}$ and by condition (*), $h(x) \neq 0$ for every $x \in \ell(f_k) \setminus Z(f)$. Moreover

$$\chi_{\ell(f_k) \setminus Z(f)} = \left(\frac{1}{h}\right)^* \cdot h \in \mathcal{J},$$

which means that $\ell(f_k) \setminus Z(f) \in J_{\mathcal{J}}$. Because

$$\{x : (\lim_{k \rightarrow \infty} (f_k - f))^*(x) \neq ((\lim_{k \rightarrow \infty} f_k)^* - f)(x)\} \subset \ell(f_k) \setminus Z(f),$$

Lemma 1 yields $(\lim_{k \rightarrow \infty} (f_k - f))^* \in \mathcal{J}$.

Simple examples show that the implications in Theorem 6 cannot be reversed.

We shall now aim at defining a density point of a set on the basis of the properties of real functions and the algebraic properties of the classes of functions distinguished here, without referring to the structure and properties of the sets under consideration. Thanks to that, it will be possible to consider, among other things, the density topology in more abstract spaces. (For example, a space in which neither topology nor measure is preassigned.) However, as will be pointed out below, our definition can be considered a generalization of both a density point in the case of measure and a J -density point in the case of category on, for example, the real line.

Let X be an arbitrary set, S be an m -ring of real functions defined on X and let \mathcal{J} be an m -ideal of S . For each $x \in X$ let us assign a set B_x such that $\chi_{B_x} \in S \setminus \mathcal{J}$ (the set B_x is called (S, \mathcal{J}) -large for x). By the symbol $B_{S, \mathcal{J}}$ we denote the family of all (S, \mathcal{J}) -large sets. For each $x \in X$ let a mapping $\varphi_x : \mathcal{A}(S) \times N \rightarrow \mathcal{A}(S)$ satisfying the following conditions:

1. $\varphi_x(A, n) \cap \varphi_x(X \setminus A, n) = \phi$, for every $A \in \mathcal{A}(S)$ and $n \in N$.
2. $\varphi_x(A \cup B, n) = \varphi_x(A, n) \cup \varphi_x(B, n)$, for every $A, B \in \mathcal{A}(S)$ such that $A \cap B = \phi$ and $n \in N$.
3. $\varphi_x(A, n) \in J_{\mathcal{J}}$, for every $A \in J_{\mathcal{J}}$ and $n \in N$.
4. $\varphi_x(A, n) \cap \varphi_x(B, n) = \varphi_x(A \cap B, n)$ for $A, B \in \mathcal{A}(S)$ and $n \in N$.
5. $\varphi_x(X, n) = X$, for $n \in N$.

By the symbol Φ_S we denote the family $\{\varphi_x : x \in X\}$.

DEFINITION. Let X be an arbitrary set, S – be an m -ring of real function defined on X and let \mathcal{J} be an m -ideal of S . Let $x_0 \in X$, B_{x_0} – (S, \mathcal{J}) -large set for

x_0 and $\varphi_{x_0} \in \Phi_S$. We say that x_0 is a $(B_{S,\mathcal{J}}, \Phi_S)$ -density point of a set $A \in \mathcal{A}(S)$, if for every strictly increasing sequence of the positive integers $\{t_k\}_{k=1}^\infty$ there exists a subsequence $\{s_{t_k}\}_{k=1}^\infty$ of $\{t_k\}_{k=1}^\infty$ such that

$$\mathcal{J} - \lim_{k \rightarrow \infty} \chi_{\varphi_{x_0}}(A, s_{t_k}) \cap B_{x_0} = \chi_{B_{x_0}}.$$

By the symbol $L_{B_{S,\mathcal{J}}, \Phi_S}(A)$ we denote the set of all $(B_{S,\mathcal{J}}, \Phi_S)$ -density points of A (if $B_{S,\mathcal{J}}, \Phi_S$ are fixed, then we short write $L(A)$).

It is not hard to verify that the above definition is a generalization of the notion of density point on the real line and I -density point in the sense of category. It is sufficient to put $B_x = [-1, 1]$ and $\phi_x(A, n) = n \cdot (A - x)$, for $x \in \mathbb{R}$ and $n \in \mathbb{N}$.

To simplify the notation, assume from now on that X is a fixed set and S is the m -ring of real functions defined on X . Let \mathcal{J} be a fixed m -ideal of S and $\mathcal{F}_{\mathcal{J}}$ be an m -system generating \mathcal{J} . Moreover, we assume that for a fixed point x_0 , B_{x_0} always denotes the set (S, \mathcal{J}) -large for x_0 and by ϕ_{x_0} we understand the function from Φ_S corresponding with x_0 .

LEMMA 2. *Let A and B be disjoint sets such that $\chi_A \in S$ and $\chi_B \in \mathcal{J}$. Then, if x_0 is a $(B_{S,\mathcal{J}}, \Phi_S)$ -density point of $A \cup B$, then x_0 is a $(B_{S,\mathcal{J}}, \Phi_S)$ -density point of A .*

Proof. From the disjointness of A and B we deduce that $\chi_{\varphi_{x_0}(A \cup B, k) \cap B_{x_0}} = \chi_{\varphi_{x_0}(A, k) \cap B_{x_0}} + \chi_{\varphi_{x_0}(B, k) \cap B_{x_0}}$ for $k \in \mathbb{N}$. Since $\chi_{\varphi_{x_0}(B, k) \cap B_{x_0}} \in \mathcal{J}$, then

$$(1) \quad \{x : \chi_{\varphi_{x_0}(A \cup B, k) \cap B_{x_0}}(x) \neq \chi_{\varphi_{x_0}(A, k) \cap B_{x_0}}(x)\} \in J_{\mathcal{J}}$$

for $k \in \mathbb{N}$.

Let $\{t_k\}$ be an arbitrary increasing sequence of positive integers and let $\{s_{t_k}\}$ be a sequence of $\{t_k\}$ such that

$$\left(\lim_{k \rightarrow \infty} \chi_{\varphi_{x_0}}(A \cup B, s_{t_k}) \cap B_{x_0}\right)^* - \chi_{B_{x_0}} \in \mathcal{J}.$$

Then, by (1), $\{x : (\lim_{k \rightarrow \infty} \chi_{\varphi_{x_0}}(A \cup B, s_{t_k}) \cap B_{x_0})^*(x) \neq (\lim_{k \rightarrow \infty} \chi_{\varphi_{x_0}}(A, s_{t_k}) \cap B_{x_0})^*(x)\} \in J_{\mathcal{J}}$ and consequently

$$\left(\lim_{k \rightarrow \infty} \chi_{\varphi_{x_0}(A, s_{t_k}) \cap B_{x_0}}\right)^* - \chi_{B_{x_0}} \in \mathcal{J},$$

which ends the proof.

LEMMA 3. *Let $A \subset B$ be sets such that $\chi_A, \chi_B \in S$. If x_0 is a $(B_{S,\mathcal{J}}, \Phi_S)$ -density point of A , then x_0 is a $(B_{S,\mathcal{J}}, \Phi_S)$ -density point of B .*

Proof. Let $\{t_k\}$ be an arbitrary increasing sequence of positive integers and $\{s_{t_k}\}$ be a subsequence of $\{t_k\}$ such that

$$\left(\lim_{k \rightarrow \infty} \chi_{\varphi_{x_0}(A, s_{t_k}) \cap B_{x_0}}\right)^* - \chi_{B_{x_0}} \in \mathcal{J}.$$

Of course, $|\lim_{k \rightarrow \infty} \chi_{\varphi_{x_0}(B, s_{t_k}) \cap B_{x_0}} - \chi_{B_{x_0}}| \leq |(\lim_{k \rightarrow \infty} \chi_{\varphi_{x_0}(A, s_{t_k}) \cap B_{x_0}})^* - \chi_{B_{x_0}}|$ and so $(\lim_{k \rightarrow \infty} \chi_{\varphi_{x_0}(B, s_{t_k}) \cap B_{x_0}})^* - \chi_{B_{x_0}} \in \mathcal{J}$, which ends the proof.

LEMMA 4. *If x_0 is a $(B_{S, \mathcal{J}}, \Phi_S)$ -density point of A , then x_0 is not a $(B_{S, \mathcal{J}}, \Phi_S)$ -density point of $X \setminus A$.*

Proof. Let $\{t_k\}$ be an arbitrary increasing sequence of positive integers and $\{s_{t_k}\}$ be a subsequence of $\{t_k\}$ such that

$$(1) \quad \left(\lim_{k \rightarrow \infty} \chi_{\varphi_{x_0}(A, s_{t_k}) \cap B_{x_0}}\right)^* - \chi_{B_{x_0}} \in \mathcal{J}.$$

Of course, $\{s_{t_k}\}$ is an increasing sequence of positive integers. Suppose, to the contrary, that x_0 is a $(B_{S, \mathcal{J}}, \Phi_S)$ -density point of $X \setminus A$. Hence there exists a subsequence $\{z_{s_{t_k}}\}$ of $\{s_{t_k}\}$ such that

$$f = \left(\lim_{k \rightarrow \infty} \chi_{\varphi_{x_0}(X \setminus A, z_{s_{t_k}}) \cap B_{x_0}}\right)^* - \chi_{B_{x_0}} \in \mathcal{J}.$$

Denote by

$$P = \{x : \lim_{k \rightarrow \infty} \chi_{\varphi_{x_0}(X \setminus A, z_{s_{t_k}}) \cap B_{x_0}}(x) \neq 1\}$$

(all points, for which above limit does not exist, belongs to P). Then $|\chi_P| \leq |f|$ and so $\chi_P \in \mathcal{J}$.

Remark that $|(\lim_{k \rightarrow \infty} \chi_{\varphi_{x_0}(A, z_{s_{t_k}}) \cap B_{x_0}})^*| \leq |\chi_P|$, consequently

$$(2) \quad \left(\lim_{k \rightarrow \infty} \chi_{\varphi_{x_0}(A, z_{s_{t_k}}) \cap B_{x_0}}\right)^* \in \mathcal{J}.$$

By (1), it is not hard to check that:

$$(3) \quad \left(\lim_{k \rightarrow \infty} \chi_{\varphi_{x_0}(A, z_{s_{t_k}}) \cap B_{x_0}}\right)^* - \chi_{B_{x_0}} \in \mathcal{J}.$$

In virtue of (2) and (3), we may infer that

$$\chi_{B_{x_0}} = \left(\lim_{k \rightarrow \infty} \chi_{\varphi_{x_0}(A, z_{s_{t_k}}) \cap B_{x_0}}\right)^* - \left[\left(\lim_{k \rightarrow \infty} \chi_{\varphi_{x_0}(A, z_{s_{t_k}}) \cap B_{x_0}}\right)^* - \chi_{B_{x_0}}\right] \in \mathcal{J}$$

which is impossible because $B_{x_0} \in B_{S, \mathcal{J}}$. This contradiction ends the proof.

It is not difficult to verify that, in the general case, the analogue of the theorem on density points does not have to hold. Therefore, in this case, it is essential to seek sufficient conditions under which such a theorem may be proved.

THEOREM 7. *Let $S, \mathcal{J}, \mathcal{F}_{\mathcal{J}}$ satisfy the following condition:*

- if $\chi_T \in S \setminus \mathcal{J}$, then there exists a function $g \in \mathcal{J}$ such that $(\chi_T + g)(X) \subset \{0, 1\}$ and for each*
- (*) *$x_0 \in (\chi_T + g)^{-1}(1) \cap T$ there exists a sequence of sets $\{T_i\} \subset \mathcal{A}(S)$ such that $B_{x_0} \subset \varphi_{x_0}(T_i, i) (i \in \mathbb{N})$ and the sequence $\{\chi_{\varphi_{x_0}(T_i, i) \setminus \varphi_{x_0}((\chi_T + g)^{-1}(1), i)}\}$ is $*$ -cofinal with $\mathcal{F}_{\mathcal{J}}$.*

Then $\chi_{A \Delta L(A)} \in \mathcal{J}$, for every $A \in \mathcal{A}(S)$. (It is not hard to verify that in the case of measure theory the assumption () is fulfilled.)*

Proof. Let $A \in \mathcal{A}(S)$. First we shall show that

$$(1) \quad \chi_{A \setminus L(A)} \in \mathcal{J}.$$

In the case if $\chi_A \in \mathcal{J}$ the condition (1) is evident. So, we may assume that $\chi_A \in S \setminus \mathcal{J}$. Let $g \in \mathcal{J}$ be a function fulfilling the condition (*).

Let us adopt $D_1 = (\chi_A + g)^{-1}(1)$ and $A_\alpha = g^{-1}(\alpha)$, where α is a value assumed by g . It is easy to see that

$$(2) \quad \chi_{A_{-1} \cap A} \in \mathcal{J}.$$

Remark that

$$(3) \quad A = (A_{-1} \cap A) \cup (D_1 \cap A).$$

If $\chi_{D_1} \in \mathcal{J}$, then $\chi_{D_1 \cap A} \in \mathcal{J}$, and consequently (according to (3)) $\chi_A = \chi_{A_{-1} \cap A} + \chi_{D_1 \cap A} \in \mathcal{J}$, which means, that (in this case) (1) is true.

Now, we assume that $\chi_{D_1} \notin \mathcal{J}$. We shall show that $D_1 \cap A \subset L(A)$. First, we remark that:

$$(4) \quad D_1 = (A_0 \cap A) \cup (A_1 \setminus A).$$

Let $x_0 \in D_1 \cap A$, then ((4)) $x_0 \in A_0 \cap A$. According to our assumptions there exists a sequence $\{T_i\} \subset \mathcal{A}(S)$ such that $B_{x_0} \subset \varphi_{x_0}(T_i, i)$ (for $i \in \mathbb{N}$) and the sequence $\{\chi_{\varphi_{x_0}(T_i, i) \setminus \varphi_{x_0}(D_1, i)}\}$ is $*$ -cofinal with $\mathcal{F}_{\mathcal{J}}$.

Let $\{p_i\}$ be an arbitrary strictly increasing sequence of positive integers. There exists a subsequence $\{s_{p_i}\}$ of $\{p_i\}$ such that $\{\chi_{\varphi_{x_0}(T_{s_{p_i}}, s_{p_i}) \setminus \varphi_{x_0}(D_1, s_{p_i})}\}$ is cofinal with $\mathcal{F}_{\mathcal{J}}$ and

$$\chi_{\varphi_{x_0}(T_{s_{p_i}}, s_{p_i}) \setminus \varphi_{x_0}(D_1, s_{p_i})} \in \mathcal{J}.$$

Then, according to (ii) of the definition of m -system,

$$f = \left(\lim_{i \rightarrow \infty} \chi_{\varphi_{x_0}(T_{s_{p_i}}, s_{p_i}) \setminus \varphi_{x_0}(D_1, s_{p_i})} \right)^* \in \mathcal{J},$$

and so

$$\chi_{\{x:f(x)\neq 0\}} \in \mathcal{J}.$$

Let us now adopt

$$Z_1 = \{x : f(x) \neq 0\} \cup \ell(\chi_{\varphi_{x_0}(T_{s_{p_i}}, s_{p_i}) \setminus \varphi_{x_0}(D_1, s_{p_i})})$$

and $Z = B_{x_0} \setminus Z_1$.

Thus $\chi_{Z_1} \in \mathcal{J}$.

It is not difficult to check that

$$\lim_{i \rightarrow \infty} \chi_{\varphi_{x_0}(T_{s_{p_i}}, s_{p_i}) \setminus \varphi_{x_0}(D_1, s_{p_i})}(x) = 0, \text{ for } x \in Z.$$

Infer that there exists $i_0 \in N$, such that for each $i \geq i_0$, $x \in \varphi_{x_0}(D_1, s_{p_i})$, which means that

$$\lim_{i \rightarrow \infty} \chi_{\varphi_{x_0}(D_1, s_{p_i})}(x) = 1, \text{ for } x \in Z.$$

Put $h = (\lim_{i \rightarrow \infty} \chi_{\varphi_{x_0}(D_1, s_{p_i}) \cap B_{x_0}})^* - \chi_{B_{x_0}}$. Hence $|h| \leq |\chi_{Z_1}|$ and so $h \in S$. Consequently $h \in \mathcal{J}$. Thus

$$\mathcal{J} - \lim_{i \rightarrow \infty} \chi_{\varphi_{x_0}(D_1, s_{p_i}) \cap B_{x_0}} = \chi_{B_{x_0}},$$

which means that $x_0 \in L(D_1)$. However, $\chi_{A_1} \in \mathcal{J}$ and so $\chi_{A_1 \setminus A} \in \mathcal{J}$ and, according to Lemma 2 (by (4)), we have that $x_0 \in L(A_0 \cap A)$. According to Lemma 3, $x_0 \in L(A)$. Thus we have really proved that

$$D_1 \cap A \subset L(A).$$

From the above and (3) we may infer that $A \setminus L(A) \subset A_{-1} \cap A$. Consequently, according to (2), we deduce that (1) is true.

We shall now prove that

$$\chi_{L(A) \setminus A} \in \mathcal{J}.$$

Infer that

$$(4) \quad L(A) \setminus A \subset (X \setminus A) \setminus L(X \setminus A).$$

Indeed, let $x \in L(A) \setminus A$. This means that $x \in X \setminus A$ and since $x \in L(A)$, by Lemma 4, $x \notin L(X \setminus A)$.

On the first part of this proof: $\chi_{(X \setminus A) \setminus L(X \setminus A)} \in \mathcal{J}$. From the above and according to (4) we have

$$|\chi_{L(A) \setminus A}| \leq |\chi_{(X \setminus A) \setminus L(X \setminus A)}|$$

and so $\chi_{L(A)\setminus A} \in \mathcal{J}$. Finally remark:

$$\chi_{A\Delta L(A)} = \chi_{A\setminus L(A)} + \chi_{L(A)\setminus A} \in \mathcal{J},$$

which ends the proof.

THEOREM 8. *Let S, \mathcal{J} satisfy the following condition:*

(**) *if $\chi_T \in S \setminus \mathcal{J}$, then there exists a function $g \in \mathcal{J}$ such that $(\chi_T + g)(X) \subset \{0, 1\}$ and for each $x_0 \in (\chi_T + g)^{-1}(1)$ there exists $k_0 \in N$ such that for every $k \geq k_0$, $B_{x_0} \subset \varphi_{x_0}((\chi_T + g)^{-1}(1), k)$.*

*Then $\chi_{A\Delta L(A)} \in \mathcal{J}$, for every $A \in \mathcal{A}(S)$. (It is not difficult to show that in the case of category (cf. [3]) the assumption (**)) is fulfilled).*

Proof. We shall first prove that $\chi_{A\setminus L(A)} \in \mathcal{J}$. If $\chi_A \in \mathcal{J}$ this fact is evident. Assume that $\chi_A \notin \mathcal{J}$. This means that there exists a function $g \in \mathcal{J}$ such that the condition (**) is fulfilled.

Let $D_1 = (\chi_A + g)^{-1}(1)$ and $A_\alpha = g^{-1}(\alpha)$, where α is a value assumed by g . It is easy to see that

$$(1) \quad \chi_{A_{-1} \cap A} \in \mathcal{J}.$$

Now, we shall show that

$$(2) \quad D_1 \cap A \subset L(A).$$

Infer that

$$(3) \quad D_1 = (A_0 \cap A) \cup (A_1 \setminus A).$$

Let now $x_0 \in D_1 \cap A$, and let k_0 be a positive integer such that $B_{x_0} \subset \varphi_{x_0}(D_1, k)$, for each $k \geq k_0$. This means that $\lim_{k \rightarrow \infty} \chi_{\varphi_{x_0}(D_1, k) \cap B_{x_0}} = \chi_{B_{x_0}}$ and so $x_0 \in L(D_1)$. Infer that $\chi_{A_1 \setminus A} \in \mathcal{J}$ and by (3) and Lemma 2, $x_0 \in L(A_0 \cap A)$, which means, according to Lemma 3, that $x_0 \in L(A)$. Thus condition (2) is true.

It is not difficult to verify that

$$A = (A_{-1} \cap A) \cup (D_1 \cap A)$$

and so ((2)) $A \setminus L(A) \subset A_{-1} \cap A$. Consequently $|\chi_{A \setminus L(A)}| \leq |\chi_{A_{-1} \cap A}|$. From the above and according to (1) we deduce that $\chi_{A \setminus L(A)} \in \mathcal{J}$.

The second part of the proof is similar to the proof of Theorem 7.

Now we present the fundamental properties of a $(B_{S, \mathcal{J}}, \Phi_S)$ -density points.

Let S be some m -ring and \mathcal{J} an m -ideal of S .

THEOREM 9. *the following conditions are fulfilled:*

- (1) *If $\chi_A, \chi_B \in S$ and $\chi_{A \Delta B} \in \mathcal{J}$, then $L(A) = L(B)$.*
- (2) *If $\chi_A, \chi_B \in S$, then $L(A \cap B) = L(A) \cap L(B)$.*
- (3) *$L(\phi) = \phi$, $L(X) = X$.*

Proof. It is evident that the conditions (1) and (3) are fulfilled.

We shall show that condition (2) takes place. Let us observe that, by Lemma 3,

$$L(A \cap B) \subset L(A) \cap L(B).$$

Let now $x_0 \in L(A) \cap L(B)$ and $\{t_k\}$ be an arbitrary strictly increasing sequence of positive integers. There exists a subsequence $\{s_{t_k}\}$ of $\{t_k\}$ such that:

$$\mathcal{J} - \lim_{k \rightarrow \infty} \chi_{\varphi_{x_0}(A, s_{t_k}) \cap B_{x_0}} = \chi_{B_{x_0}}$$

and there exists a subsequence $\{z_{s_{t_k}}\}$ of $\{s_{t_k}\}$ such that

$$\mathcal{J} - \lim_{k \rightarrow \infty} \chi_{\varphi_{x_0}(B, z_{s_{t_k}}) \cap B_{x_0}} = \chi_{B_{x_0}}.$$

It is not difficult to see that

$$\mathcal{J} - \lim_{k \rightarrow \infty} \chi_{\varphi_{x_0}(A, z_{s_{t_k}}) \cap B_{x_0}} = \chi_{B_{x_0}}.$$

Put

$$A' = \{x : \lim_{k \rightarrow \infty} \chi_{\varphi_{x_0}(A, z_{s_{t_k}}) \cap B_{x_0}}(x) = 1\}$$

and

$$B' = \{x : \lim_{k \rightarrow \infty} \chi_{\varphi_{x_0}(B, z_{s_{t_k}}) \cap B_{x_0}}(x) = 1\}.$$

Hence $B_{x_0} \setminus (A' \cap B') \in J_{\mathcal{J}}$, which implies that

$$B_{x_0} \setminus \{x : \lim_{k \rightarrow \infty} \chi_{\varphi_{x_0}(A, z_{s_{t_k}}) \cap \varphi_{x_0}(B, z_{s_{t_k}}) \cap B_{x_0}}(x) = 1\} \in J_{\mathcal{J}}$$

and so

$$\left(\lim_{k \rightarrow \infty} \chi_{\varphi_{x_0}(A \cap B, z_{s_{t_k}}) \cap B_{x_0}} \right)^* - \chi_{B_{x_0}} \in \mathcal{J}.$$

Consequently, from the arbitrary choice of $\{t_k\}$, we may infer that $x_0 \in L(A \cap B)$.

The theory built so far allows one to expect that relatively simple constructions will lead to the building of some topology. However, in order to make its creation possible, it is necessary to introduce additional assumptions. To avoid dispensable complications, we shall accept, as our assumption, a comparatively simple (and self-evident) condition – up to the end of the article we shall constantly assume (this assumption also concerns the case of the notion of a “weak density point” introduced further) that we consider exclusively spaces X , operators Φ_S (and ψ_S), m -algebras S , m -ideals \mathcal{J} and families $B_{S,\mathcal{J}}$, such that the following condition is satisfied:

if $\{A_\lambda\}_{\lambda \in \Lambda}$ is a family of sets such that $\chi_{A_\lambda} \in S$ and $A_\lambda \subset L_{B_{S,\mathcal{J}},\Phi_S}(A)$ for each $\lambda \in \Lambda$, then $\chi_{\bigcup_{\lambda \in \Lambda} A_\lambda} \in S$.

THEOREM 10. *Let X be an arbitrary set. Then the family*

$$\mathcal{T} = \{A \in \mathcal{A}(S) : A \subset L_{B_{S,\mathcal{J}},\Phi_S}(A)\}$$

is a topology in X . (We say that \mathcal{T} is a $(B_{S,\mathcal{J}},\Phi_S)$ -density topology).

We say that the density theorem takes place if $\chi_{A \Delta L_{B_{S,\mathcal{J}},\Phi_S}(A)} \in \mathcal{J}$, for every $A \in \mathcal{A}(S)$.

LEMMA 5. *If the density theorem takes place, then $\chi_{L_{B_{S,\mathcal{J}},\Phi_S}(A)} \in S$, for every A such that $\chi_A \in S$.*

Proof. Since $A \cap L_{B_{S,\mathcal{J}},\Phi_S}(A) = A \setminus (A \setminus L_{B_{S,\mathcal{J}},\Phi_S}(A))$ and $\chi_{A \setminus L_{B_{S,\mathcal{J}},\Phi_S}(A)} \in \mathcal{J} \subset S$ then $\chi_{A \cap L_{B_{S,\mathcal{J}},\Phi_S}(A)} \in S$. Consequently

$$\chi_{L_{B_{S,\mathcal{J}},\Phi_S}(A)} = \chi_{L_{B_{S,\mathcal{J}},\Phi_S}(A) \setminus A} + \chi_{A \cap L_{B_{S,\mathcal{J}},\Phi_S}(A)} \in S.$$

THEOREM 11. *If the density theorem takes place, then V is open in a $(B_{S,\mathcal{J}},\Phi_S)$ -density topology if and only if $V = L_{B_{S,\mathcal{J}},\Phi_S}(A) \setminus B$, for some A and B such that $\chi_A \in S$ and $\chi_B \in \mathcal{J}$.*

The theory presented above gives a method to define density points in more abstract spaces. We have examples applying this theory in the case of second countable m -dimensional manifolds. The methods of construction are connected

with topology, measure theory and algebra. The density topologies obtained possess many interesting properties. Since the constructions are long, we omit these examples.

Many of the considerations (but not all) are true if instead of condition 2 of the definition of the family Φ_S we write:

- 2'. If $A, B \in \mathcal{A}(S)$ and $A \subset B$, then $\varphi_x(A, n) \subset \varphi_x(B, n)$ for $n = 1, 2, \dots$ ($x \in X$) and if C is a set such that $\chi_C \in \mathcal{J}$, then $\varphi_x(X \setminus C, n) = X \setminus \varphi_x(C, n)$ for $n = 1, 2, \dots$ ($x \in X$).

The family Φ_S for which condition 2' is fulfilled, we shall denote by ψ_S . We say that a $(B_{S, \mathcal{J}}, \psi_S)$ -density point is a weak density point (if $B_{S, \mathcal{J}}, \psi_S$ are fixed). Let us adopt as $L_w(A)$ the set of all weak density points of the set A . Then $L_w : \mathcal{A}(S) \rightarrow 2^X$ we shall call weak density operator.

We shall now investigate some properties of these objects.

LEMMA 6. *The operator L_w possesses the following fundamental properties:*

- (1) *If $\chi_A, \chi_B \in S$ and $A \subset B$, then $L_w(A) \subset L_w(B)$.*
- (2) *If $\chi_A, \chi_B \in S$, then $L_w(A \cap B) = L_w(A) \cap L_w(B)$.*
- (3) *$L_w(\phi) = \phi$ and $L_w(X) = X$.*

THEOREM 12. *Let X be an arbitrary set. Then the family*

$$\mathcal{T}_w = \{A \in \mathcal{A}(S) : A \subset L_w(A)\}$$

is a topology in X . (We call \mathcal{T}_w the weak density topology.)

Remark. Since every family Φ_S fulfills condition 2', every $(B_{S, \mathcal{J}}, \Phi_S)$ -density topology is a weak density topology.

THEOREM 13. *Every set A such that $\chi_A \in \mathcal{J}$ is boundary and closed in a weak density topology.*

Proof. We shall first prove that A is boundary. If $A = \phi$, then this fact is obvious. Let $A \neq \phi$. We shall show that any non-empty open set, in a weak density topology, does not include in A . Indeed, let $\phi \neq B \subset A$. Then $B \in J_{\mathcal{J}}$. This means that

- (1) $\varphi_x(B, n) \cap B_x \in J_{\mathcal{J}}$, for every $x \in X$ and $n \in N$

To prove the boundary of A it suffices to demonstrate that $L_w(B) = \phi$.

Let $x_0 \in X$ and let $\{t_k\}$ be an arbitrary strictly increasing sequence of positive integers and $\{s_{t_k}\}$ a subsequence of $\{t_k\}$. According to (1) we have

$$\left(\lim_{n \rightarrow \infty} \chi_{\varphi_{x_0}(B, n) \cap B_{x_0}}\right)^* \in \mathcal{J}.$$

If $x_0 \in L_w(B)$ then $\mathcal{J} - \lim_{n \rightarrow \infty} \chi_{\varphi_{x_0}(B, n) \cap B_{x_0}} = \chi_{B_{x_0}}$ and so $\chi_{B_{x_0}} \in \mathcal{J}$ which is impossible, because B_{x_0} is (S, \mathcal{J}) -large set. The obtained contradiction proves that $L_w(B) = \phi$.

To finish, let us notice that $X \setminus A \in \mathcal{T}_w$, i.e.

$$(2) \quad X \setminus A \subset L_w(X \setminus A).$$

In fact, let $x \in X \setminus A$ and $\{t_k\}$ be a strictly increasing sequence of positive integers. Assume that $\{s_{t_k}\} = \{t_k\}$. Thus

$$\varphi_x(X \setminus A, s_{t_k}) \cap B_x = B_x \setminus \varphi_x(A, s_{t_k}), \text{ for every } k \in N.$$

From the above equality it follows that

$$B_x \setminus \bigcup_k \varphi_x(A, s_{t_k}) \subset \{z : \lim_{k \rightarrow \infty} \chi_{\varphi_x(X \setminus A, s_{t_k}) \cap B_x}(z) = 1\}$$

and so

$$\left| \left(\lim_{k \rightarrow \infty} \chi_{\varphi_x(X \setminus A, s_{t_k}) \cap B_x}\right)^* - \chi_{B_x} \right| \leq \left| \chi_{\bigcup_k \varphi_x(A, s_{t_k})} \right|.$$

This means that $x \in L_w(X \setminus A)$ and so from the arbitrary choice of x we obtain (2).

COROLLARY 1. *If \mathcal{J} includes all characteristic functions of singletons, then (X, \mathcal{T}_w) is a T_1 -space.*

COROLLARY 2. *If \mathcal{J} includes all characteristic functions of singletons, then (X, \mathcal{T}_w) is not a separable space (we assume that in X there exists at least one (S, \mathcal{J}) -large set).*

In monograph [1] and in the paper [7] a lower density operator and an abstract density operator are considered.

The following theorem is true with the additional assumption that the σ -algebra contains all singletons.

THEOREM 14. *Let X be an arbitrary set. Then every lower density operator is a weak density operator. (Precisely, for a lower density operator L there exists S, \mathcal{J}, ψ_S and $B_{S, \mathcal{J}}$ such that $L = L_w$).*

Proof. Suppose that Σ is a σ -algebra of subsets of X and $L : \Sigma \rightarrow \Sigma$ is a lower density operator. Let S be the family of all functions measurable with respect to Σ . (Then $\mathcal{A}(S) = \Sigma$). Moreover let $\mathcal{J} = \{\text{const}_0\}$ be the m -ideal of the m -ring S . Let us adopt $B_x = \{x\}$ as a set (S, \mathcal{J}) -large (for $x \in X$). Put $\varphi_x(A, n) = L(A)$, for $x \in X$, $A \in \mathcal{A}(S)$ and $n \in N$.

Thus $\varphi_x : \mathcal{A}(S) \times N \rightarrow \mathcal{A}(S)$ and φ_x fulfills the conditions 1, 2', 3, 4, 5 for ψ_S .

Now, we shall show that for every $A \in \mathcal{A}(S) = \Sigma$

$$(1) \quad L(A) = L_w(A).$$

Indeed, let $x \in L(A)$. Then $\varphi_x(A, n) \cap B_x = \{x\}$, for $n = 1, 2, \dots$. Consider an arbitrary strictly increasing sequence $\{t_k\}$ of positive integers and put $\{s_{t_k}\} = \{t_k\}$.

Then

$$\lim_{k \rightarrow \infty} \chi_{\varphi_x(A, s_{t_k}) \cap B_x} = \chi_{B_x}$$

and so

$$\mathcal{J} - \lim_{k \rightarrow \infty} \chi_{\varphi_x(A, s_{t_k}) \cap B_x} = \chi_{B_x},$$

which means that $x \in L_w(A)$.

Now let $x \in L_w(A)$ and assume $x \notin L(A)$. Let $\{t_k\}$ be an arbitrary sequence, strictly increasing of positive integers and $\{s_{t_k}\}$ an arbitrary subsequence of $\{t_k\}$. Hence

$$\chi_{\varphi_x(A, s_{t_k}) \cap B_x} = \text{const}_0 \text{ (for } k = 1, 2, \dots),$$

this means that

$$\left(\lim_{k \rightarrow \infty} \chi_{\varphi_x(A, s_{t_k}) \cap B_x} \right)^* - \chi_{B_x} = -\chi_{\{x\}} \notin \mathcal{J};$$

which contradicts the fact that $x \in L_w(A)$. This contradiction shows that (1) is true and consequently the theorem has been proved.

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