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Heredity of Density Points[2]

For $t \in (0,1]$ let $\Omega_t^n =$

$$\{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : \text{ for each } i \in \mathbb{N} x_i \geq 0 \text{ and } \|x\| = \left(\sum x_i^2\right)^{1/2} \leq t\}$$

and let g be a Lebesgue measurable real function defined on $\Omega^n = \Omega_1^n$, strictly positive almost everywhere and with the property that $g(rx) \leq g(x)$ if $r < 1$. We assume further that g is bounded and $g(x) \to 0$ as $x \to 0$. Let $\mathcal{F}_n$ denote the set of such functions. If $S \subset \Omega^n$, then $S \cap \Omega_t^n$ is denoted by S_t. Let A be a Lebesgue measurable subset of Ω^n such that 0 is a density point of A; that is, $\frac{|A_t|}{|\Omega_t^n|} \to 1$ as $t \to 0$, where $|\Omega_t^n| = K_n t^n$. Our aim is to study the density of A at 0 with respect to the measure $\mu_g(E) = \int_E g$. We show that generally 0 is not a density point of A with respect to μ_g in the usual sense. If g in $\mathcal{F}_n$ satisfies an additional condition, always valid in the case $n = 1$, then we prove that 0 is an upper μ_g-density point of A; that is, $\limsup \frac{\mu_g(A_t)}{\mu_g(\Omega_t^n)} = 1$ as $t \to 0$. Stronger and necessary conditions on both A and g for 0 to be a μ_g-density point of A are examined.

Given a function $g \in \mathcal{F}_n$ and r, $s \in (0,1)$, we let $M(g,r,s) =$ ess sup $g(x)$ and $m(g,r,s) =$ ess inf $g(x)$ where $x \in \Omega_s^n \setminus \Omega_r^n$.

Theorem 1 *Let 0 be a density point of a set $A \subset \Omega^n$ and let $g \in \mathcal{F}_n$. Assume further that there is an $\alpha > 0$ and a strictly decreasing sequence $\{t_k\}$ with $t_1 = 1$ and $t_k \to 0$ such that $\frac{M_k}{m_k} \leq \alpha$, where $M_k = M(g, t_{k+1}, t_k)$ and $m_k = m(t, t_{k+1}, t_k)$. Then 0 is an upper density point of A with respect to μ_g.*

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[2]Dedicated to the memory of Jorge Smith

Proof. If the Theorem is not true, then there is an $\epsilon > 0$ and a j such that $\mu_g(\Omega_t^n \setminus A) > \epsilon\mu_g(\Omega_t^n)$ for all $t \in (0, t_j]$. Therefore for $k > j$

$$\alpha|\Omega_{t_k}^n \setminus A| = \alpha \sum_{h=k}^{\infty} |\Omega_{t_h}^n \setminus \Omega_{t_{h+1}}^n \setminus A| \geq \sum_{h=k}^{\infty} \int_{\Omega_{t_h}^n \setminus \Omega_{t_{h+1}}^n \setminus A} \frac{g}{m_h} =$$

$$\int_{\Omega_{t_k} \setminus A} \frac{g}{m_k} + \sum_{h=k+1}^{\infty} \int_{\Omega_{t_h}^n \setminus A} g \left(\frac{1}{m_h} - \frac{1}{m_{h-1}} \right) \geq$$

$$\epsilon \frac{\mu_g(\Omega_{t_k}^n)}{m_k} + \epsilon \sum_{h=k+1}^{\infty} \mu_g(\Omega_{t_h}^n) \left(\frac{1}{m_h} - \frac{1}{m_{h-1}} \right) =$$

$$\epsilon \sum_{h=k}^{\infty} \frac{\mu_g \left(\Omega_{t_h}^n \setminus \Omega_{t_{h+1}}^n \right)}{m_h} \geq \epsilon \sum_{h=k}^{\infty} |\Omega_{t_h}^n \setminus \Omega_{t_{h+1}}^n| = \epsilon |\Omega_{t_k}^n|$$

which is a contradiction. □

The following example shows that the additional condition on g is an essential hypotheses in Theorem 1 for $n \geq 2$.

Example.

Let $t_k = \frac{1}{k}$ and $A(k) = \{re^{i\theta} : r \in (0,1],\ \theta \in [0, \frac{\pi}{2} - \frac{1}{2^k}]\} \cap \left[\Omega_{t_k}^2 \setminus \Omega_{t^{k+1}}^2\right]$, and let $A = \cup_{k=1}^{\infty} A(k)$. It follows immediately that 0 is a density point of A. Let $B = \Omega^2 \setminus A$ and $B(k) = B \cap \left[\Omega_{t_k}^2 \setminus \Omega_{t_{k+1}}^2\right]$. We define

$$g(x) = \begin{cases} \frac{1}{k} & \text{if } x \in B(k) \\ \\ \frac{B(k)}{2k|A(k)|} & \text{if } x \in A(k) \end{cases}.$$

It is clear that $g \in \mathcal{F}_2$ and $\mu_g(A_t) < \mu_g(B_t)$ for any $t \in (0, 1)$.

On the other hand the condition of Theorem 1 is fulfilled in the one dimensional case. Indeed take $t_1 = 1,\ t_2 = \inf\{x \in \Omega^1 : \frac{g(t_1^-)}{2} < g(x) \leq g(t_1^-)\}$ and generally $t_{k+1} = \inf\{x \in \Omega^1 : \frac{g(t_k^-)}{2} < g(x) \leq g(t_k^-)\} < t_k$. Since $g(t_k^-) \leq \frac{g(t_1^-)}{2^{k-1}}$, it follows that $g(t_k) \to 0$ as $k \to \infty$ whence due to the form of g, $t_k \to 0$ as $k \to \infty$. Besides $\frac{M_k}{m_k} \leq 2$ for any k.

Theorem 2 *Let 0 be a density point of $A \subset \Omega^n$. A necessary and sufficient condition for 0 to be a μ_g-density point of A for any $g \in \mathcal{F}_n$ is that $|A_t| = |\Omega_t^n|$ for t small enough.*

Proof. The condition is obvious sufficient. To prove the necessary part, assume that $|B_t| > 0$ for all t, where $B = \Omega^n \setminus A$. Then B_t has density points in the interior of Ω^n_t. Let V_1 be a (r_1, t_1)-radial neighborhood of a density point of B in the interior of Ω^n_1; that is, if $x \in V_1$, then $r_1 < \|x\| < t_1$ and $\lambda x \in V_1$, $\frac{r_1}{\|x\|} < \lambda < \frac{t_1}{\|x\|}$, where $0 < r_1 < t_1 < 1$. It is easy to see that we can also take V_1 in such a way that $|B \cap V_1| - |A \cap V_1| = c_1 > 0$, $|V_1| < |\Omega^n_{t_1} \setminus \Omega^n_{r_1} \setminus V_1|$. We now apply a recurrent procedure to construct radial neighborhoods V_i assuming that $V_1, \ldots, V_{i-1}$ are given, $i = 2, \ldots$. For $s_i = \min\left(\frac{1}{i}, r_{i-1}\right)$ let V_i be a (r_i, t_i)-radial neighborhood of a density point of B in the interior of $\Omega^n_{s_i}$ where $t_i \le s_i$ and $|B \cap V_i| - |A \cap V_i| = c_i > 0$, $|V_i| < |\Omega^n_{t_i} \setminus \Omega^n_{r_i} \setminus V_i|$. Next a function $g \in \mathcal{F}_n$ will be defined. For $x \in \left(\Omega^n_1 \setminus \Omega^n_{t_1}\right) \cup V_1$ let $g(x) = 1$. For $x \in \left(\Omega^n_{t_i} \setminus \Omega^n_{t_{i+1}} \setminus V_i\right) \cup V_{i+1}$, for $i = 1, 2, \ldots$, let $g(x) = \frac{\left(\prod_{j=1}^{i} c_j\right)}{2^i} \prod_{j=1}^{i} \left|\left(\Omega^n_{t_j} \setminus \Omega^n_{t_{j+1}} \setminus V_j\right)\right| < \frac{1}{2^i}$. So it is easy to see that the function g is in $\mathcal{F}_n$ and to see that $\mu_g\left(A \cap \left[\Omega^n_{t_i} \setminus \Omega^n_{t_{i+1}}\right]\right) < \mu_g\left(B \cap \left[\Omega^n_{t_i} \setminus \Omega^n_{t_{i+1}}\right]\right)$ for all i. Hence 0 is not a μ_g-density point of A.□

A function $g \in \mathcal{F}_n$ is said to be of slow increase at the origin if there exists an $\alpha > 0$ and a $\lambda \in (0,1)$ such that $\frac{M(g,\lambda t,t)}{m(g,\lambda t,t)} \le \alpha$ for any $t \in (0,1]$.

Theorem 3 *a) Let g be a function of slow increase at the origin. Then for any measurable set A with density 1 at the origin, 0 is a μ_g-density point of A.*
b) In the one dimensional case the condition of slow increase at the origin is also necessary to obtain this result.

Proof. of a) Let $\lambda \in (0,1)$. Since A has density 1 at 0, for t small enough $|A \cap [\Omega^n_t \setminus \Omega^n_{\lambda t}]| \ge \frac{|\Omega^n_t \setminus \Omega^n_{\lambda t}|}{2}$. Thus

$$\mu_g(A_t) \ge (1 - \lambda^n)|\Omega^n_t| \frac{m(g, \lambda t, t)}{2}.$$

On the other hand, since $g(rx) \le g(x)$ for all $r \in (0,1)$, ess sup $g(x) \le M(g, \lambda t, t)$ for $x \in \Omega^n_t \setminus A$ and hence

$$\mu_g(\Omega^n_t \setminus A) \le |\Omega^n_t \setminus A| M(g, \lambda t, t),$$

whence $\frac{\mu_g(\Omega^n_t \setminus A)}{\mu_g(A_t)} \to 0$ as $t \to 0$. □

Proof. of b) Let $\lambda_\nu = \frac{\nu}{\nu+1}$ for $\nu = 1, 2, \dots$. If g does not satisfy the condition of slow increase at the origin, then we can obtain, by an iterative procedure, a sequence $\{x_\nu\}$ with $x_1 = 1$, $0 < x_{\nu+1} < \frac{x_\nu}{2}$ and $\frac{g(x_{\nu+1})}{g(\lambda_\nu x_{\nu+1})} \geq \nu$. Let

$$\delta_\nu = \lambda_\nu x_{\nu+1} \frac{g(\lambda_\nu x_{\nu+1})}{g(x_{\nu+1})} + x_{\nu+1}(1 - \lambda_\nu) \leq 2\frac{x_{\nu+1}}{\nu+1}.$$

Then

$$\mu_g([x_{\nu+1}, x_{\nu+1} + \delta_\nu]) \geq g(x_{\nu+1})\delta_\nu \geq$$
$$\mu_g([0, \lambda_\nu x_{\nu+1}]) + \mu_g([\lambda_\nu x_{\nu+1}, x_{\nu+1}]) = \mu_g([0, x_{\nu+1}]).$$

Let $B = \cup_{\nu=1}^{\infty} [x_{\nu+1}, x_{\nu+1} + \delta_\nu]$. By the above estimate B does not have density 0 at the origin with respect to μ_g. On the other hand we show now that B has density 0 at the origin. Indeed for $\nu \geq k$ we have $\delta_\nu \leq \frac{2}{k+1}\frac{x_{k+1}}{2^{\nu-k}}$. Then

$$\sum_{\nu=k}^{\infty} \delta_\nu \leq \frac{4x_{k+1}}{k+1}.$$

Pick $t \in (0, 1)$. If $x_{k+1} + \delta_k \leq t \leq x_k$, then

$$\frac{|B_t|}{t} = \frac{\sum_{\nu=k}^{\infty} \delta_\nu}{t} < \frac{4}{k+1}.$$

If $x_{k+1} < t < x_{k+1} + \delta_k$, then

$$\frac{|B_t|}{t} < \frac{\sum_{\nu=k}^{\infty} \delta_\nu}{t} < \frac{4}{k+1},$$

whence $\frac{|B_t|}{t} \to 0$ as $t \to 0$.□

For $n \geq 2$ we introduce polar coordinates (r, x') to express the points x of $\Omega^n \setminus \{0\}$. We set $r = \|x\|$, and $x' = \frac{x}{r}$. Let $\Sigma = \{y \in \Omega^n : \|y\| = 1\}$.

Theorem 4 *Let $g \in \mathcal{F}_n$ and set $\phi(r) = \int_\Sigma g(rx')\,d\sigma$. Then $\phi \in \mathcal{F}_1$. Moreover ϕ is of slow increase at the origin if for each measurable set A with density 1 at the origin, 0 is a μ_g-density point of A.*

Proof. It is easy to show that $\phi \in \mathcal{F}_1$. Assume that $\phi(r)$ is not of slow increase at the origin. Then neither is the function (in $\mathcal{F}_1$) $r^{n-1}\phi(r)$, whence by Theorem 3 b) there is a set B in Ω^1 with density 0 at the origin such that

$$\frac{\int_{B_t} r^{n-1}\phi(r)\,dr}{\int_0^t r^{n-1}\phi(r)\,dr} \not\to 0 \text{ as } t \to 0.$$

If $\tilde{B} = \{x \in \Omega^n : \|x\| = r \text{ for some } r \in B\}$, then $|\tilde{B}_t| = \omega_{n-1}\mu_{r^{n-1}}(B_t)$, where ω_{n-1} is the surface area of Σ. Since the function r^{n-1} is of slow increase at the origin, it follows from Theorem 3 a) that

$$\frac{|\tilde{B}_t|}{|\Omega_t|} = \frac{\mu_{r^{n-1}}(B_t)}{\mu_{r^{n-1}}([0,t])} \to 0 \text{ as } t \to 0.$$

On the other hand $\frac{\mu_g(\tilde{B}_t)}{\mu_g(\Omega_t^n)} = \frac{\int_{B_t} r^{n-1}\phi(r)\,dr}{\int_0^t r^{n-1}\phi(r)\,dr}$ does not tend to 0 as $t \to 0$.$\square$.

References

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