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ON TWO GENERALIZATIONS OF THE DARBOUX PROPERTY

The study of those real functions which are the sum of a continuous function and a Darboux function has inspired two natural extensions of the Darboux property which have already been investigated in some aspects by Ellis [EL], Massera [MA], Radakovič [RD], Bruckner, Ceder and Weiss [BCW], etc. From the characterizations given in [BCW] the two classes of real functions seem to be related. We shall discuss some topics concerning this relationship, as well as others.

0. Preliminaries.

Definition 0. Let $I \subset \mathbf{R}$ be an interval (or $\mathbf{R}$ itself) and let $f : I \rightarrow \mathbf{R}$. For $x \in I$ let:

$$C_0(f,x) = \{y \in \overline{\mathbf{R}} : \forall\, N \in V(y), \forall\, M \in V(x)\ \ f^{-1}(N) \cap M \neq \phi\}$$

$$C(f,x) = \{y \in \overline{\mathbf{R}} : \forall\, N \in V(y), \forall\, M \in V(x) \text{ card } (f^{-1}(N) \cap M) = c\}$$

where $V(y)$ is the set of the neighborhoods of y.

The one sided cluster sets $C_0^-(f,x)$, $C_0^+(f,x)$, $C^-(f,x)$, $C^+(f,x)$ are defined similarly, working with one sided neighborhoods M.

A function $f : I \rightarrow \mathbf{R}$ is called generalized Darboux if for every interval $J \subset I$ the set $f(J)$ is dense in the interval $[\inf\{f(x) : x \in J\}, \sup\{f(x) : x \in J\}]$ and uniformly generalized Darboux if for every interval $J \subset I$ and every set A of cardinality less than c the set $f(J \setminus A)$ is dense in $[\inf\{f(x) : x \in J\}; \sup\{f(x) : x \in j\}]$.

As in [BCW], the classes of generalized (respectively uniformly generalized) Darboux functions will be denoted by U_0 (respectively U). In [BCW] it is shown that U is the closure of D, the class of Darboux functions, under uniform convergence. Also the following two lemmas are proved which yield complete characterizations for the classes U and U_0:

Lemma 1. *Let $f : I \rightarrow \mathbf{R}$. The following conditions are equivalent:*

(i) $f \in U_0$

(ii) *any one sided cluster set* $C_0^\varepsilon(f,x)$, $\varepsilon \in \{\pm\}$ *is a closed interval (in* $\mathbf{R}$*).*

(iii) *for any* $a, b \in I$, $a < b$ *we have:*

$$\bigcup\{C_0(f^*,x) : x[a,b]\} = [\inf\{f(x) : x \in [a,b]\}, \sup\{f(x) : x \in [a,b]\}].$$

where $f^* = f|[a,b]$

Lemma 2. *Under the same assumptions the following assertions are equivalent:*

(i) $f \in U$

(ii) *for any* $a, b \in I$, $a < b$ *we have*

$$\bigcup\{C(f^*,x) : x \in [a,b]\} = [\inf\{f(x) : x \in [a,b]\}, \sup\{f(x) : x \in [a,b]\}]$$

(iii) $f \in U_0$ *and* f *(i.e., the graph of* f*) is c-dense in itself*

(iv) $f \in U_0$ *and for any open* N, $f^{-1}(N)$ *is empty or c-dense in itself.*

1. Results.

A. A multiplicative analogue of a theorem of Sierpinski.

A classical result due to Sierpinski asserts that every real function is the sum of two Darboux functions. However, as Solomon Marcus has pointed out in [MS 1] there exists a real function which is not the product of two Darboux functions. Things are different with respect to the class U as is seen from the following:

Theorem 1. *Any function* $f : \mathbf{R} \to \mathbf{R}$ *is the product of two uniform generalized Darboux functions.*

Proof: Let us consider the partition induced on $\mathbf{R}$ by congruence modulo Q (i.e. $x \equiv y$ iff $x - y \in Q$). Let $k = \mathbf{R}/\equiv$. Since each equivalence class is countable, (assuming the Continuum Hypothesis) k is of cardinality c. Hence we may write

$$\mathbf{R} = \bigcup\{C_\lambda : \lambda \in \mathbf{R}\} \tag{1}$$

where $a : \mathbf{R} \to k$ is a bijection and we write C_λ for $a(\lambda)$. We consider a partition $k = L \cup H$, $L \cap H = \phi$ with $L \sim H \sim (\mathbf{R}\backslash\{0\})$. Let $b : \mathbf{R}\backslash\{0\} \to a^{-1}(L)$ and

$c : \mathbf{R}\backslash\{0\} \to a^{-1}(H)$ be bijections. (Their existence is guaranteed by the fact that a is bijective.) Set

$$g(x) = \begin{cases} r & \text{if } x \in C_{b(r)} \\ f(x)/s & \text{if } x \in C_{c(s)} \end{cases} \quad \text{and} \quad h(x) = \begin{cases} f(x)/r & \text{if } x \in C_{b(r)} \\ s & \text{if } x \in C_{c(s)} \end{cases}$$

for any $f, s \in \mathbf{R}\backslash\{0\}$. Clearly g and h are well defined and $g \cdot h = f$. Since both classes $C_{b(r)}$ and $C_{c(s)}$ are dense in $\mathbf{R}$, g and h take every nonnull real value in every interval. It follows immediately that $g, h \in U$.

B. Preconnectivities and generalized Darboux functions.

Throughout this paragraph we will identify a function and its graph. We recall the following definitions: $f : X \to Y$ is said to be connected provided its graph is a connected subset of $X \times Y$; f is a connectivity function provided that, whenever C is a connected subset of $X, f|C$ is connected (X and Y are two topological spaces). It is well known that a function $f : I \to \mathbf{R}$ (where I is connected) is a connectivity function iff it is connected.

Definition 1. $A \subset X$ is called preconnected iff $\overline{A}$ is connected.

Definition 2. A function $f : X \to Y$ is said to be preconnected if its graph is a preconnected subset of $X \times Y$.

Definition 3. A function $f : X \to Y$ is called a preconnectivity function provided, whenever C is a connected subset of X, $f|C$ is preconnected.

There is no analogue to the result quoted before (namely preconnectedness does not imply preconnectivity) as the following example shows. Let

$$f(x) = \begin{cases} \sin 1/x & \text{if } x > 0 \\ 1 & \text{if } x = 0 \\ 0 & \text{if } x < 0. \end{cases}$$

The following result offers characterizations of preconnectivity functions:

Theorem 2. *Let A be a nontrivial connected subset of* $\mathbf{R}$ *and* $f : A \to \mathbf{R}$. *Then the following are equivalent:*

(i) $f \in U_0$

(ii) *f is a preconnectivity function.*

Proof: (i) $\to$ (ii) Consider $[a,b] \subset A$, $a < b$, and take $f^* = f|[a,b]$. It is readily seen that (even for an arbitrary function) we have

(2) $$\overline{\Gamma}_{f^*} = \{(x,y) : x \in \overline{A} \text{ and } y \in C_0(f^*,x)\}.$$

Let us consider the decomposition of $\overline{\Gamma}_{f^*}$ into connected components.

(3) $$\overline{\Gamma}_{f^*} = \bigcup\{C_i : i \in I\}.$$

Since projection on the x axis, π_1, is continuous, the sets $G_i = \pi_1(C_i) \subset \mathbf{R}$, are connected. Let us show that each G_i is closed. Let $x_0 \in \overline{G}_i$. Suppose, to the contrary that $x_0 \notin G_i$ and take $(x_n)_{n=1}^{\infty} \subset G_i$ with $x_n \to x_0$. Without loss of generality we may suppose (since $x_0 \neq x_n$, $\forall\, n \geq 1$) that $(x_n)_{n=1}^{\infty}$ is strictly monotone (for example, strictly increasing). Since $x_1, x_n \in G_i$ and since G_i is connected (hence convex), $[x_1,x_n] \subset G_i$ $\forall\, n \geq 1$ and hence $[x_1,x_0) = \bigcup_{n\geq 1}[x_1,x_n] \subset G_i$. It follows that $\{(x,f(x)) : x \in [x_1,x_0)\} \subset C_i$. Since C_i is closed (being a connected component) and since $f \in U_0$, $(x_0,f(x_0)) \in C_i$. (f being in U_0, we may find a strictly increasing sequence $(y_n)_{n=1}^{\infty} \subset [x_1,x_0)$, $y_n \to x_0$ and $f(y_n) \to f(x_0)$. Consequently $x_0 \in G_i$.) Thus G_i is closed.

Finally, for any $y_0 \in \overline{A}$, the points of the set $\{y_0\} \times C_0(f,y_0)$ lie in a single connected component. Indeed $C_0(f,y_0)$ is either $C_0^-(f,y_0)$, $C_0^+(f,y_0)$ or $C_0^-(f,y_0) \cup C_0^+(f,y_0)$. Since $f \in U_0$, $C_0^-(f,y_0)$ and $C_0^+(f,y_0)$ are closed intervals containing $f(y_0)$. So, in any case, $C_0(f,y_0)$ is connected. It follows that $\{y_0\} \times C_0(f,y_0)$ is connected. Hence all its points lie in a single connected component. This means that for $i \neq j$ we have $G_i \cap G_j = \phi$. Since $\overline{A}$ is connected, since $\overline{A} = \bigcup\{G_i : i \in I\}$ and since the sets are closed an mutually disjoint, it follows that card $(I) = 1$; i.e., $\overline{\Gamma}_{f^*}$ connected.

(ii) $\to$ (i) We will show that f satisfies condition (iii) of **Lemma 1**. Take $a < b \in A$ and set $f_* = f|[a,b]$. Then $\overline{\Gamma}_{f_*} = \{(x,y) : y \in C_0(f_*,x),\ x \in [a,b]\}$. Denoting the projection of $\overline{\Gamma}_{f_*}$ on the y-axis by B, we have

(4) $$B = \bigcup\ C_0(f_*,x) : x \in [a,b]\}.$$

Since $\overline{\Gamma}_{f_*}$ is connected and since projection on the y-axis is a continuous function, B is connected and hence convex. It follows easily that $\inf f_*(x)$, $\sup f_*(x)$ (x in $[a,b]$) are in B. (For example, let us show that $\inf\{f(x) : x \in [a,b]\} \in \mathbf{R}$. For any $n \in \mathbf{N}$ take $x_n \in [a,b]$ such that $|f_*(x_n) - \inf\{f(x) : x \in [a,b]\}| \leq 1/n$. The sequence $(x_n)_{n=1}^{\infty}$ has an accumulation point $x_0 \in [a,b]$. Hence $\inf\{f(x) : x \in [a,b] \in C_0(f_*,x_0) \subset B$.) It follows that $[\inf f(x), \sup f(x)] : (x \in [a,b]) \subset \bigcup\{C_0(f_*,x) : x \in [a,b]\}$. The other inclusion is obvious. Hence $\forall\, a < b \in A \bigcup\{C_0(f_*,x) : x \in [a,b]\} = [\inf f_*(x), \sup f_*(x)]$ which means that $f \in U_0$.

C. Quasiuniform Arzelá-Gagaeff-Alexandrov Convergence.

It is known that the classes U_0 and U are closed under uniform convergence. It is natural to ask if this statement is valid when we replace uniform convergence by a weaker type of convergence. Moreover, since the classes U_0 and U are "related", it would be interesting to find a type of convergence so that, under this type of convergence, the closure of D is exactly U_0. (We recall that the closure of D under uniform convergence is U.) In what follows we offer a first glimpse, by studying the behavior of U and U_0 under quasiuniform Arzelá-Gagaeff-Alexandrov convergence.

Definition 4. Let (X, ρ) and (Y, σ) be two metric spaces. The sequence $(f_n)_{n=1}^{\infty}, f_n : X \to Y$ converges quasiuniformly to f in the sense of Arzelá-Gagaeff-Alexandrov provided that f_n converges pointwise to f and, for each $\varepsilon > 0$, there is a sequence (not necessarily infinite) of positive integers $n_1 < n_2 \ldots < n_p < n_{p+1} \ldots$ and a corresponding sequence of open subsets of X, $G_1, G_x \ldots G_p, G_{p+1} \ldots$ such that $X = \bigcup\{G_i : i \geq 1\}$ and, for every $i \geq 1$ and $x \in G_i$, we have $\sigma(f(x), f_{n_i}(x)) < \varepsilon$.

Clearly uniform convergence implies quasiuniform Arzelá-Gagaeff-Alexandrov convergence.

Theorem 3. *The class U_0 is closed with respect to quasiuniform Arzelá-Gagaeff- Alexandrov (abbreviated AGA) convergence.*

Proof: Let $(f_n)_{n=1}^{\infty}$ be a sequence in U_0 with $f_n \overset{AGA}{\longrightarrow} f$. Let $x_0 \in I$. We will prove that each cluster set $C_0^-(f, x_0)$, $C_0^+(f, x_0)$ is connected (and then we will apply **Lemma 1**). Let us prove $C_0^-(f, x_0)$ is connected. Let $\alpha = \inf C_0^-(f, x_0)$ and $\beta = \sup C_0^-(f, x_0)$. If $\alpha = \beta$, there is nothing to prove. If $\alpha < \beta$, let $z \in (\alpha, \beta)$ and set $\gamma = \min\{|z - \alpha|, |z - \beta|\}$. Take $n \geq 1$. Since $f_n \overset{AGA}{\longrightarrow} f$, we may find G_{i_n} an open neighborhood of x_0 such that for any $x \in G_{i_n}$

$$(*)|f_{i_n}(x) - f(x)| < \frac{1}{3}\min(\gamma, 1/n).$$

For $n \in \mathbf{N}$ let $\alpha_n = \inf C_0^-(f_{i_n}, x_0)$ and $\beta_n = \sup C_0^-(f_{i_n}, x_0)$. Let us prove that $z \in (\alpha_n, \beta_n)$. Since $\alpha = \inf C_0^-(f, x_0)$, by the Definition 0, it is easy to see that there exists a sequence $(x_n^{(n)})_{m=1}^{\infty}$ such that

(i) $x_m^{(n)} \in G_{i_n}$, $\forall\, m \geq 1$

(ii) $x_m^{(n)} < x_0$, $\forall\, m \geq 1$

(iii) $x_m^{(n)} \to x_0\, (m \to \infty)$ and $f(x_m^{(n)}) \to \alpha$.

Since $|f(x_m^{(n)}) - f_{i_n}(x_m^{(n)})| \leq \frac{1}{3}\min(\gamma, 1/n)$ and $f(x_m^{(n)}) \to \alpha$, it follows easy (provided $\alpha \in \mathbf{R}$) that the sequence $(f_{i_n}(x_m^{(n)})_{m\geq 1}$ is bounded. Hence we may find a convergent subsequence $(f_{i_n}(x_{m_k}^{(n)}))_{k=1}^{\infty}$. Let u be its limit. It is clear that $u \geq \alpha_n$. On the other hand from $(*)$ it follows that $u \leq \alpha + \frac{1}{3}\min(\gamma, 1/n)$. Hence $\alpha_n \leq \alpha + \frac{1}{3}\min(\gamma, 1/n)$. By the definition of γ, $\alpha_n < z$. If $\alpha = -\infty$, then (as is easily verified) $\alpha_n = -\infty$. (The case $\alpha = +\infty$ is not possible). We assumed that $-\infty + c = -\infty$.

Analogously one proves that $z < \beta_n$. Hence $z \in (\alpha_n, \beta_n)$. Since $f_{i_n} \in U_0$, we may select $y_n \in G_{i_n}$ with $y_n < x_0$ such that $|y_n - x_0| < 1/m$ and $|f_{i_n}(y_n) - z| < 1/n$. It follows easily that $y_n \to x_0$ and that $f_{i_n}(y_n) \to z$ $(n \to \infty)$. Since $|f_{i_n}(y_n) - f(y_n)| < \frac{1}{3n}$, it follows that $\lim_n f(y_n) = z$. Thus $z \in C_0^-(f, x_0)$. But $z \in (\alpha, \beta)$ was arbitrary. Hence $C_0^-(f, x_0)$ (and analogously $C_0^+(f, x_0)$) is connected. By **Lemma 1** it follows that $f \in U_0$. q.e.d.

The corresponding result for uniformly generalized Darboux functions is as follows:

Theorem 4. *The class U is closed under quasiuniform Arzelá-Gagaeff-Alexandrov convergence.*

Proof: Let $f_n \in U$ with $f_n \overset{AGA}{\longrightarrow} f$ $(n \to \infty)$. Since $f_n \in U \subset U_0$, by **Theorem 3** $f \in U_0$. Let N be an open interval such that $f^{-1}(N) \neq \phi$, let $x_0 \in f^{-1}(N)$ and set $y_0 = f(x_0)$. Since x_0 was chosen arbitrarily in $f^{-1}(N)$, to prove that $f^{-1}(N)$ is c-dense in itself, it suffices to show that for any neighborhood V of x_0 card $(f^{-1}(N) \cap V) = c$. Consider such a neighborhood V. Moreover set $\alpha = \inf N$ and $\beta = \sup N$. Then $N = (\alpha, \beta)$. Put $\gamma = \frac{1}{2}\min\{|\alpha - y_0|, |\beta - y_0|, 1\}$, let $M = (y_0 - \gamma, y_0 + \gamma)$, and let $\varepsilon = \gamma/3$. There is an index $n_k \in \mathbf{N}$ and an open neighborhood $V_{n_k} \in V(x_0)$ such that $\forall\ x \in V_{n_k}$ we have $|f(x) - f_{n_k}(x)| < \delta$. Then $f_{n_k}(x_0) \in M$, and hence $f_{n_k}^{-1}(M)$ is c-dense in itself. Show that $V \cap V_{n_k} \cap f_{n_k}^{-1}(M) \subset V \cap f^{-1}(N)$. Let $z \in f_{n_k}^{-1}(M) \cap V \cap V_{n_k}$. Then $|f_{n_k}(z) - y_0| < \gamma$. Since $|f_{n_k}(z) - f(z)| < \gamma/3$, $|f(z) - y_0| < \gamma + \gamma/3 < 2\gamma$. Thus $z \in f^{-1}(N)$ and hence $z \in f^{-1}(N) \cap V$. Since $V \cap V_{n_k}$ is an open neighborhood of x_0 and $x_0 \in f_{n_k}^{-1}(M)$, card $(V \cap V_{n_k} \cap f_{n_k}^{-1}(M)) = c$. Hence card $(V \cap f^{-1}(N)) = c$. It follows that $f^{-1}(N)$ is c-dense in itself. Since $f \in U_0$, by **Lemma 2**, $f \in U$. q.e.d.

Remark: Since $D \subset U$, (denoting by D^{AGA} the closure of the class of Darboux function under AGA convergence) $\overline{D} \subset D^{AGA} \subset U^{AGA}$. (The first inclusion holds because uniform convergence implies AGA convergence.) Since $\overline{D} = U^{AGA} = U$, $D^{AGA} = U$. Hence we fail to "reach" U_0 from D by AGA convergence. During the revising of this paper we have found a type of convergence such that $\overline{D} = U_0$.

D. Stationary sets for generalized Darboux (uniformly generalized Darboux) functions.

Throughout this paragraph all the functions are defined on $(0,1)$. Following Boboc and Marcus [BM] we say that $A \subset (0,1)$ is stationary for a class F of functions if any $f \in F$ is constant on $(0,1)$ when it is constant on A. In the same paper it is shown that the set A is stationary for the class D iff its complement is at most countable. (To obtain this result the Continuum Hypothesis is assumed.) Here we characterize the stationary sets for the classes U and U_0. We shall see assume the Continuum Hypothesis.

To attain this goal we must recall the following:

Lemma 3 [BM]. *A real set contains a c-dense in itself subset iff it is of cardinality c.*

With these preliminaries we can prove the next result.

Theorem 5. $A \subset (0,1)$ *is stationary for the class U iff its complement is at most countable.*

Proof: (i) The condition is necessary: Suppose that A is stationary for the class U. Since $D \subset U$, A is stationary for the class D. By the results in [BM] it follows that its complement is at most countable.

Remark: Since $D \subset U \subset U_0$, the condition is still necessary when replacing U by U_0.

(ii) The condition is sufficient: Assume that A is not stationary for the class U. We will show that C_A has the power of continuum. We may assume without loss of generality that A is dense in $(0,1)$ (otherwise C_A would contain an interval). Since A is not stationary for U, there is a constant k and a nonconstant function $f \in U$ such that $f(x) = k \; \forall \, x \in A$. Let

$$B = \{x \in (0,1) : f(x) \neq k\}.$$

Clearly $B \subset C_A$. We will show that B is c-dense in itself. Let $x \in B$. Assume that $f(x) < k$ (to make a choice) and let I be an interval containing x. Since A is dense in $(0,1)$, let $y \in I$ such that $f(y) = k$. From the fact that $f \in U$ it follows that $C = \{z \in [x,y] : f(z) \in (f(x), f(y))\}$ is uncountable (otherwise it would contradict the definition of U). Since $C \subset B \cap [x,y] \subset B \cap I$, $B \cap I$ is uncountable. Hence B is c-dense in itself. By **Lemma 3** C_A is uncountable.

Let us pass to the study of the stationary sets for the class U_0.

Theorem 6. *$A \subset (0,1)$ is stationary for the class U_0 iff its complement does not contain a set which is bilaterally dense in itself.*

Proof: (i) The condition is necessary: Let A be stationary for U_0. It follows that its complement is at most countable. Since the case when $(0,1)\backslash A$ is finite is trivial, suppose $(a_n)_{n\geq 0}$ is an enumeration of $(0,1)\backslash A$. Let $(r_n)_{n=0}^{\infty}$ be an enumeration of $\mathbf{Q}\backslash\{0\}$. Suppose to the contrary that $(0,1)\backslash A$ contains a countable subset B bilaterally dense in itself. We will construct a nonnull function $f : (0,1) \to \mathbf{R}$, $f \in U_0$ such that $f = 0$ on A. For any $x \in (0,1)\backslash B$ (hence also in A) put $f(x) = 0$. Let $(b_n)_{n=0}^{\infty}$ be an enumeration of B. We will define f inductively:

Step $n = 0$. Put $f(b_0) = 1$. Since B is bilaterally dense in itself, we may consider two subsequences of (b_n), $(b^0_{i_k})_{k=0}^{\infty}$ and $(b^0_{j_k})_{k=0}^{\infty}$ such that for any $k \geq 0$, $b^0_{i_k} < b_0 < b^0_{j_k}$ and $\lim_k b^0_{i_k} = \lim_k b^0_{j_k} = b_0$. Put

$$f(b^0_{i_k}) = f(b^0_{j_k}) = \begin{cases} r_n & \text{iff } k \text{ is a power of } p_n \text{ the n-th prime number} \\ 1 & \text{otherwise.} \end{cases}$$

Step $n+1$. Suppose $f(b_0), \ldots, f(b_n)$ have been already defined and for any $0 \leq \ell \leq n$ we have chosen two sequences $(b^\ell_{i_k})_{k=0}^{\infty}$ and $(b^\ell_{j_k})_{k=0}^{\infty}$ just as in Step $n = 0$.

If $f(b_{n+1})$ has not yet been defined, put $f(b_{n+1}) = 1$. Consider (as in Step $n = 0$) two subsequences: $(b^{n+1}_{i_k})_{k=0}^{\infty}$ and $(b^{n+1}_{j_k})_{k=0}^{\infty}$ such that for any $k \geq 0$

$$b^{n+1}_{i_k} < b_{n+1} < b^{n+1}_{j_k}$$

and

$$\lim_k b^{n+1}_{i_k} = \lim_k b^{n+1}_{j_k} = b_{n+1}.$$

Furthermore, just as for any $0 \leq \ell \leq n$, $\lim_k b^\ell_{i_k} = \lim_k b^\ell_{j_k} = b_\ell \neq b_{n+1}$, we may suppose that for no $k \geq 0$ have $f(b^{n+1}_{i_k})$ and $f(b^{n+1}_{j_k})$ been defined (by choosing them in an adequate neighborhood of b_{n+1}). Put

$$f(b^{n+1}_{i_k}) = f(b^{n+1}_{j_k}) = \begin{cases} r_t & \text{if } k \text{ is a power of } p_t \\ 1 & \text{otherwise.} \end{cases}$$

We have constructed $f : (0,1) \to \mathbf{R}$ such that $f = 0$ on A. Moreover, since $\lim_k b^n_{i_{p_t^k}} = \lim_k b^n_{j_{p_t^k}} = b_n$ and for any k $f(b^n_{i_{p_t^k}}) = f(b^n_{j_{p_t^k}}) = r_t$ and $\mathbf{Q}\backslash\{0\}$ is dense in $\mathbf{R}$, it is not hard to show that $f \in U_0$.

(ii) The condition is sufficient: Suppose A is not stationary for U_0. Then we may find $k \in \mathbf{R}$ and $f \in U_0$ such that $f = k$ on A but $f \neq k$. Let $B = \{x \in (0,1) : f(x) \neq k\}$. Since $f \in U_0$, B is bilaterally dense in itself. But $B \subset (0,1)\backslash A$. q.e.d.

E. Another characterization for the class U_0.

We have seen that generalized Darboux functions do not preserve connectedness but have some "weaker" remarkable properties. In the sequel we will characterize them as the functions that "almost" preserve connectedness (in a sense it is given below).

For $A \subset \mathbf{R}$ let

$$\nu(A) = \sup\{\lambda(B) : B = B^0,\ B \subset co\,(A)\backslash A\}$$

where B^0 is the interior of B, λ is Lebesgue measure, and $co(A)$ is the convex hull of A. $\nu(A)$ is a reasonable "measure" of the "holes" of the set A. Let $f : \mathbf{R} \to \mathbf{R}$ and let $I_0 \subset \mathbf{R}$ be an interval. Put

$$\omega_f^\nu(I_0) = \sup\{\nu(f(I) : I \subset I_0,\ I \text{ interval}\}$$

and

$$\omega_f^\nu(x_0) = \inf\{\omega_f^\nu(I) :\ I \text{ interval},\ x_0 \in I^0\}.$$

$\omega_f^\nu(x_0)$ is called the ν-oscillation of f at x_0.

Finally, say that f is ν-continuous at x_0 iff $\omega_f^\nu(x_0) = 0$. It is easy to see that every generalized Darboux function defined on an interval I_0 is ν-continuous. (Indeed, for any interval $I \subset I_0$, $co\ f(I) = [\inf\{f(x) : x \in I\}, \sup\{f(x) : x \in I\}]$. Hence the set $f(I)$ is dense in $co\,(f(I))$. Thus $\nu(f(I)) = 0$.

Lemma 4. *If $\omega_f^\nu(x_0) = 0$, then the cluster sets $C_0^-(f, x_0)$ and $C_0^+(f, x_0)$ are connected.*

Proof: We shall prove the assertion only for $C_0^-(f, x_0)$. Suppose that $C_0^-(f, x_0)$ is not connected. Since $C_0^-(f, x_0)$ is closed, there are $y_0 \in co\,(C_0^-(f, x_0))$ and $\varepsilon > 0$ such that $(y_0 - \varepsilon,\ y_0 + \varepsilon) \subset co\,(C_0^-(f, x_0))$ and $(y_0 - \varepsilon,\ y_0 + \varepsilon) \cap C_0^-(f, x_0) = \phi$. Then we find $b < x_0$ such that $\forall\ t \in (b, x_0),\ f(t) \notin [y_0 - \varepsilon/2,\ y_0 + \varepsilon/2]$. Since $y_0 + \varepsilon \in co\,(C_0^-(f, x_0)) = [\inf C_0^-(f, x_0), \sup C_0^-(f, x_0)]$, there are $x', y' \in C_0^-(f, x_0)$ with $x' > y_0 + \varepsilon/2$ and $y' < y_0 - \varepsilon/2$. Set $\alpha = \min\{|x' - y_0 - \varepsilon/2|, |y' - y_0 + \varepsilon/2|\}$. Let $p \le b$ and $V = [p, x_0]$. Since $x', y' \in C_0^-(f, x_0)$, there are $x_1, y_1 \in V$ with $|f(x_1) - x'| < \alpha/2$ and $|f(y_1) - y'| < \alpha/2$. It follows easily that $f(y_1) < y_0 - \varepsilon/2 < y_0 + \varepsilon/2 < f(x_1)$. Consequently $f(V)$ contains the "hole" $[y_0 - \varepsilon/2, y_0 + \varepsilon/2]$. Hence $\nu(f(V)) \ge \varepsilon$.

Let I be an interval containing x_0 as an interior point. $\nu(f(I \cap V)) \ge \varepsilon$ implies $\omega_f^\nu(x_0) \ge \varepsilon$. But this is a contradiction since $\omega_f^\nu(x_0) = 0$. Hence $C_0^-(f, x_0)$ is connected.

Theorem 7. *Let $f : \mathbf{R} \to \mathbf{R}$ be a real function. Then f is ν-continuous iff f is generalized Darboux.*

Proof: The assertion follows directly from **Lemma 4** and **Lemma 1**.

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