RESEARCH ARTICLES Real Analysis Exchange Vol. 17 (1991-92)
Ali A. Alikhani-Koopaei, Pennsylvania State University, Berks Campus, Reading, PA 19610-6009.

BOREL MEASURABILITY OF EXTREME LOCAL DERIVATIVES

1 Introduction.

The extreme generalized derivative has been studied by a number of mathematicians. For example for the extreme bilateral derivatives by Banach [3], Kempisty [8], and Hajek [7] whose successive results are that these derivatives are measurable, in Baire class 3, and in Baire class two. Misik [9], [10] proved that for any ordinal number α the upper (lower) unilateral approximate derivatives of Borel functions of the class α are lower (upper) semi-Borel functions of the class $\alpha + 2$. In [1], [2] we studied the Borel measurability of extreme path derivatives when the system of paths is continuous and non-porous, respectively. The notion of local systems introduced by Thomson [12] is the most appropriate setting for extreme generalized derivatives.

It would be most interesting to have general versions of the above mentioned results to help explain how the similarities and the differences arise. In this paper the Borel measurability of local systems is defined and it is shown that for a unilateral local system S with an overlapping property, the Borel measurability of S is a necessary condition for Borel measurability of $\overline{F}'_S(x)$. We also use the Borel measurability of local systems to study the nature of non-removable S-discontinuities of a bounded function.

2 Preliminaries.

Throughout the paper all sets are subsets of the real line, $\mathbf{R}$, and $\mathbf{N}$ denotes the set of positive integers. The complement of A in $\mathbf{R}$ is denoted by $A^{\sim}$. The sets of the Borel additive class α, Borel multiplicative class α are denoted by AB_α, MB_α, respectively.

We first state the following definitions from B.S. Thomson [11] and A.M. Bruckner [4].

2.1 Definition. A local system is a family S such that at each point $x \in \mathbf{R}$ there is given a nonempty collection of sets with the following properties:

(i) $\{x\} \notin S(x)$, for all x,
(ii) if $s_1 \in S(x)$ and $s_1 \subseteq s_2$, then $s_2 \in S(x)$,
(iii) if $s \in S(x)$, then $x \in s$,
(iv) if $s \in S(x)$ and $\delta > 0$, then $s \cap (x - \delta, x + \delta) \in S(x)$.

2.2 Definition. Let $S = \{S(x) : x \in \mathbf{R}\}$ be a local system.

2.2.1 S is said to be filtering at a point $x \in \mathbf{R}$, if $s_1 \cap s_2 \in S(x)$ whenever s_1 and s_2 belong to $S(x)$.
2.2.2 S is said to be overlapping at a point $x \in \mathbf{R}$, if $s_1 \cap s_2 \neq \{x\}$ whenever s_1 and s_2 belong to $S(x)$.

If S has any of these properties at each point, then we say that S has that property.

2.3 Definition. Let $S = \{S(x) : x \in \mathbf{R}\}$ be a local system and let $F : \mathbf{R} \to \mathbf{R}$. The function F is said to be S-continuous at the point x, provided that for every $\epsilon > 0$ the set $\{t : | F(x) - F(t) | < \epsilon\} \in S(x)$.

2.4 Definition. Let S be a local system. The dual of S denoted by S^* is the local system defined so that, for $x \in \mathbf{R}$, $s \in S^*(x)$ if and only if
(i) $x \in s$, and
(ii)$(\mathbf{R} \setminus s) \cup \{x\} \notin S(x)$.

2.5 Definition. Let S be a local system and let $A \subseteq \mathbf{R}$. The set $S-der(A) = \{x \in \mathbf{R} : A \cup \{x\} \in S(x)\}$ is called the derived set of A for the local system S.

2.6 Definition. For any real function F and any local system S, the function

$$\underline{F}'_S(x) = \sup_{\sigma \in S(x)} \inf \left\{ \frac{F(y) - F(x)}{y - x} : y \in \sigma, y \neq x \right\}$$

is called the lower extreme local derivative of F with respect to S. Similarly $\overline{F}'_S(x)$ is defined. From the definition of extreme local derivatives it follows that $\overline{F}'_S = \underline{F}'_{S^*}$ when S and S^* are dual.

Bruckner, O'Malley and Thomson [6] introduced the concept of path derivative. A path at x is a set $E_x \subseteq \mathbf{R}$ such that $x \in E_x$ and x is a point of accumulation of E_x. Let $E = \{E_x : x \in \mathbf{R}\}$ be a system of paths. The path derivative and extreme path derivatives are the limit and limit superior (limit inferior) of the quotient $\frac{F(y)-F(x)}{y-x}$ as y tends to x, $y \in E_x$, respectively. One is interested in finding a local system S such that $\overline{F}'_S = \overline{F}'_E$ for any function F.

2.7 **Definition**. Let $E = \{E_x : x \in \mathbf{R}\}$ be a system of paths. The local system $S = \{S(x) : x \in \mathbf{R}\}$ is generated by the system of paths E if for every $x \in \mathbf{R}$, the family $S(x)$ is the filter generated by a filter base of the form $\{E_x \cap (x - \eta, x + \eta) : \eta > 0\}$.

Let α be an ordinal number and let $F : \mathbf{R} \to \mathbf{R}$. Then F is a lower (upper) semi-Borel function of the class α if and only if the sets $\{x : F(x) > \beta\}$ ($\{x : F(x) < \beta\}$) are of the Borel additive class α for all $\beta \in \mathbf{R}$.

The closed sets are said to be of type F_0. The unions of denumerably many sets of type F_0 are called of type F_1. The intersections of denumerably many sets of type F_1 are called of type F_2. It is clear that this process can be continued to generate a class of sets, F_α, corresponding to each ordinal number α. The set A is of type G_α if $A^\sim$ is of type F_α.

3 Borel measurability (measurability) of local systems.

3.1 **Definition**. Let $S = \{S(x) : x \in \mathbf{R}\}$ be a local system.

(i) S is said to be of type F_α (of type G_α) if the set $S - der(A)$ is of type F_α (of type G_α), for every Borel measurable set $A \subseteq \mathbf{R}$.

(ii) S is said to be Borel measurable if S is of type F_α or G_α for some ordinal number α.

(iii) S is said to be Lebesgue measurable if $S - der(A)$ is Lebesgue measurable for every Lebesgue measurable set $A \subseteq \mathbf{R}$.

(iv) S is said to be of type AB_α (of type MB_α) if $S - der(A)$ is of type AB_α (of type MB_α) for every arbitrary set $A \subseteq \mathbf{R}$.

In this section we investigate the Borel measurabilty of known local systems, using the same notations as [11].

3.2 **Example.** Let $A \subseteq \mathbf{R}$. Then $S_0 - der(A)$ is the set of all points x for which $A \cup \{x\}$ is a neighborhood of x. Consequently this derived set is easily seen to be open. Thus the local system S_0 is of type G_0. Similarly the local systems S_0^+ and S_0^- are both of type F_1.

3.3 **Example.** Let $S_{ess}^+ = \{S_{ess}^+(x) : x \in \mathbf{R}\}$ and $S_{ess}^- = \{S_{ess}^-(x) : x \in \mathbf{R}\}$ be the local systems defined as $\sigma \in S_{ess}^+(x)$ if and only if $x \in \sigma$ and such that σ has right hand exterior, upper density positive at x, and $\sigma \in S_{ess}^-(x)$ if and only if $x \in \sigma$ and σ has left hand exterior, upper density positive at x. From lemma 23.10 of [11] follows that S_{ess}^+ and S_{ess}^- are both local systems of type G_2.

3.4 **Lemma.** Let S be a local system and let S^* be its dual. Then S is of the type F_α (measurable or G_α) if and only if S^* is of the type G_α (measurable or F_α).

Proof: Let A be a Borel measurable subset of $\mathbf{R}$. Then

$$\begin{aligned} S^* - der(A) &= \{x : A \cup \{x\} \in S^*(x)\} \\ &= \{x : (A \cup \{x\}) \bigcap \sigma \neq \{x\} \text{ for all } \sigma \in S(x)\} \\ &= \{x : (\mathbf{R} \setminus A) \cup \{x\} \notin S(x)\} \\ &= (S - der(\mathbf{R} \setminus A))^{\sim}. \end{aligned}$$

Hence $S^* - der(A)$ is a set of the type G_α or measurable if and only if $S - der(\mathbf{R} \setminus A)$ is of type F_α or measurable, respectively.

3.5 **Example.** From example 3.3, it is clear that the local system S_1 described by the requirement that $\sigma \in S_1(x)$ if and only if $x \in \sigma$ and either $\overline{d}_+^e(\sigma, x) > 0$ or $\overline{d}_-^e(\sigma, x) > 0$, is of type G_2. Theorem 12.3 of [11] shows that the local systems S_1 and $S_{ap}(\sigma \in S_{ap}(x)$ if and only if $x \in \sigma$ and $\underline{d}^i(\sigma, x) = 1)$ are dual. Thus the local system S_{ap} is of type F_2.

3.6 **Lemma.** Let $A \subseteq \mathbf{R}$, and define the real function F by

$$F(x) = P_+(A \cup \{x\}, x) = \limsup_{h \to 0+} \frac{\lambda(A \cup \{x\}, x, h)}{h}$$

where $\lambda(A \cup \{x\}, x, h)$ is the length of the largest interval contained in $(A \cup \{x\})^\sim \cap (x, x+h)$. Then $F(x)$ is the right hand porosity of A at any point x and F is a lower semi-Borel function of the class 2.

Proof: Let $r \in \mathbf{R}$. For $r < 0$ or $r \geq 1$ the set $\{x : F(x) > r\}$ is the entire real line or the empty set, respectively. Thus it is trivially a $G_{\delta\sigma}$ set.

Let $0 \leq r < 1$ and $\{r_i\}_{i=1}^\infty$ be an enumeration of rational numbers in [0,1]. Then we will show that $\{x : F(x) > r\} = \bigcup_{i=1}^\infty \{x : \exists$ a sequence of open intervals $\{I_n\} \subseteq (A \cup \{x\})^\sim, I_n$ contracting to x, and $| I_n | [\frac{1-r-r_i}{r+r_i}] > d(x, I_n)\}$. Here $| I_n |$ and $d(x, I_n)$ are the length of I_n and the distance of x to I_n, respectively. Let $A_i = \{x : \exists$ a sequence of open intervals $\{I_n\} \subseteq (A \cup \{x\})^\sim, I_n$ contracting to x, and $| I_n | [\frac{1-r-r_i}{r+r_i}] > d(x, I_n)\}$. If $y \in \{x : F(x) > r\}$, then $F(y) > r, i.e. \limsup_{h\to 0+} \frac{\lambda(A\cup\{y\},y,h)}{h} > r$. So there is a sequence of open intervals $\{I_n\} \subseteq (A \cup \{y\})^\sim$, and I_n contracts to y so that $\lim_{n\to\infty} \frac{|I_n|}{d(y,I_n)+|I_n|} > r$. Hence there is a rational number r_i in (0,1) such that $\lim_{n\to\infty} \frac{|I_n|}{d(y,I_n)+|I_n|} > r + r_i$. Without loss of generality we can assume that $\frac{|I_n|}{d(y,I_n)+|I_n|} > r + r_i$ for all n. Thus $| I_n |> (r + r_i)[d(y, I_n)+ | I_n |]$. Hence $| I_n | [\frac{1-r-r_i}{r+r_i}] > d(y, I_n)$ and finally $y \in \bigcup_{i=1}^\infty A_i$.

On the other hand let $y \in \bigcup_{i=1}^\infty A_i$. Then there exists a sequence $\{I_n\} \subset (A\cup\{y\})^\sim$ such that $\{I_n\}$ contracts to y and $| I_n | [\frac{1-r-r_i}{r+r_i}] > d(y, I_n)$ for some $r_i \in$(0,1), implying $\frac{|I_n|}{d(y,I_n)+|I_n|} > r + r_i$. Therefore $\limsup_{h\to 0+} \frac{\lambda(A\cup\{y\},y,h)}{h} > r + r_i > r$. Now the set A_i is shown to be G_δ. The set $A_i^\sim = \bigcup_{m=2}^\infty F_{m,i}$ where,$F_{m,i} = \{x :$ for every open interval $I \subset [x, x+1/m] \cap A^\sim, | I | [\frac{1-r-r_i}{r+r_i}] \leq d(x, I)\}$. We show $F_{m,i}$ is a closed set for each m.

Let $x_k \in F_{m,i}$ and $x_k \to z$. Since $x_k \in F_{m,i}$ for$I \subset A^\sim \cap [x_k, x_k + 1/m]$, we have $| I | [\frac{1-r-r_i}{r+r_i}] \leq d(x_k, I)$. If $z \notin F_{m,i}$, then there exists an $I \subseteq [z, z + 1/m] \cap A^\sim$ so that $| I | [\frac{1-r-r_i}{r+r_i}] > d(z, I)$. Hence $d(z, I)- | I | [\frac{1-r-r_i}{r+r_i}] < -\epsilon - \frac{(r+r_i)}{1-r-r_i}\epsilon$ for some $0 < \epsilon < 1$. Let J be an open interval such that the left end point of J is the same as the left end point of I and J is a proper subset of I with $\frac{1-r-r_i}{r+r_i}(| I | - | J |) < \epsilon/2$. Choose K large enough so that $| x_k - z |< \epsilon/2$ and $J \subset [x_k, x_k + 1/m] \cap A^\sim$. Then

$$d(x_k, J)- \mid J \mid [\tfrac{1-r-r_i}{r+r_i}] = d(x_k, J) - d(z, J) + d(z, J)$$
$$-d(z, I) + d(z, I) - \tfrac{1-r-r_i}{r+r_i} \mid I \mid + \tfrac{1-r-r_i}{r+r_i}(\mid I \mid - \mid J \mid)$$
$$< \epsilon/2 + \epsilon/2[\tfrac{r+r_i}{1-r-r_i}] + [-\epsilon - \tfrac{r+r_i}{1-r-r_i}\epsilon] < 0.$$

Thus $d(x_k, J) <\mid J \mid [\frac{1-r-r_i}{r+r_i}]$, which it implies that $x_k \notin F_{m,i}$, leading to a contradiction. Therefore each $F_{m,i}$ is a closed set and $\{x : F(x) > r\} = \bigcup_{i=1}^{\infty} \bigcap_{m=1}^{\infty} F^{\sim}_{m,i}$ is a $G_{\delta\sigma}$ set.

3.7 **Corollary**. For $0 \leq t < 1$, let PS_t denote the local system defined at each point x as:
$PS_t(x) = \{S : x \in S$, and the porosity of S at x is $\leq t\}$,
$PS_1 = \{S : x \in S$, and the porosity of S at x is $< 1\}$.
From lemma 3.6 it follows that the local systems PS_t, for $0 \leq t < 1$ and PS_1 are of type F_2 and F_3 respectively.

4 Applications of Borel measurability of local systems.

Let F be a function, $E = \{E_x : x \in \mathbf{R}\}$ be a system of paths and let S be the local system generated by E. Then the following lemma shows that $\overline{F}'_S = \overline{F}'_E$. Therefore some of the results in [1] and [2] could be expressed in this setting with respect to a path generated local system.

4.1 **Lemma**. Let F be a function and S be the local system generated by the system of paths $E = \{E_x : x \in \mathbf{R}\}$. Then $\overline{F}'_S = \overline{F}'_E$.

Proof: It is clear that $\overline{F}'_E \geq \overline{F}'_S$, since for each $x \in \mathbf{R}$

$$\begin{aligned} \overline{F}'_E(x) &= \limsup_{y \to x,\, y \in E_x} \tfrac{F(y)-F(x)}{y-x} \\ &= \inf_{n \in \mathbf{N}} \sup \left\{ \tfrac{F(y)-F(x)}{y-x} : y \in E_x \cap (x - 1/n, x + 1/n) \right\}. \end{aligned}$$

On the other hand, if $\overline{F}'_S(x) = -\infty$, then for each $n \in \mathbf{N}$, we have a $\sigma_n \in S(x)$ such that $\sup\left\{\frac{F(y)-F(x)}{y-x} : y \in \sigma_n, y \neq x\right\} < -n$. But $\sigma_n \supseteq (E_x \bigcap (x - \eta_n, x + \eta_n))$ for some $\eta_n > 0$. Hence

$$\begin{aligned}\overline{F}'_E(x) &= \inf_{n\in\mathbf{N}} \sup\{\tfrac{F(y)-F(x)}{y-x} : y \in E_x \bigcap (x-1/n, x+1/n),\ y \neq x\} \\ &\leq \inf\{(-n) : n \in \mathbf{N}\} = -\infty.\end{aligned}$$

When $\overline{F}'_E(x)$ is finite, using the infimum property, it is easy to show that $\overline{F}'_E(x) \leq \overline{F}'_S(x)$,so $\overline{F}'_E(x) = \overline{F}'_S(x)$.

4.2 **Definition**. A local system $S = \{S(x) : x \in \mathbf{R}\}$ is called right unilateral (left unilateral) if for each $x \in \mathbf{R}$ and for all $s \in S(x)$ we have $s \cap [x, \infty] \in S(x)$ (for $s \in S(x)$ we have $s \cap (-\infty, x] \in S(x)$).

The theorem below shows that the Borel measurability (measurability) of a local system is a necessary condition for Borel measurability (measurability) of local derivates of a Borel measurable (measurable) function, when the local system is left (right) unilateral and has the overlapping property.

4.3 **Theorem**. Let S be a right unilateral local system with overlapping property. If $\overline{F}'_S(x)$ is Borel measurable (measurable) for any given Borel measurable (measurable) function F, then the local system S is necessarily Borel measurable (measurable).

Proof: Suppose S is not a Borel measurable (measurable)local system. Then there is a Borel measurable (measurable) set A so that $S - der(A)$ is not a Borel measurable (measurable) set. Let $F(x) = -\chi_A(x)$. Then clearly F is Borel measurable (measurable).

I: For $x \in A \cap (S - der(A)), \overline{F}'_S(x) = 0$.

For each $x \in \mathbf{R}$ there is a sequence $\{\sigma_{x,n}\}_{n=1}^{\infty} \in S(x)$ so that

$$\overline{F}'_S(x) = \limsup_{\to\infty}\{\frac{F(y)-F(x)}{y-x} : y \in \sigma_{x,n}, y \neq x\}.$$

When $x \in A \cap (S - der(A))$ then $A = A \cup \{x\} \in S(x)$. Thus

$$\begin{aligned}\overline{F}'_S(x) &\geq \limsup_{n\to\infty}\{\tfrac{F(y)-F(x)}{y-x} : y \in \sigma_{x,n} \cap A,\ y \neq x\} \\ &\geq \limsup_{n\to\infty}\{\tfrac{-1-(-1)}{y-x} : y \in \sigma_{x,n} \cap A\} = 0.\end{aligned}$$

On the other hand

$$\begin{aligned}\overline{F}'_S(x) &= \inf_{\sigma\in S(x)}\sup\{\tfrac{F(y)-F(x)}{y-x} : y\in\sigma,\, y\neq x\}\\ &\leq \sup\{\tfrac{F(y)-F(x)}{y-x} : y\in A, y\neq x\} = 0.\end{aligned}$$

So $\overline{F}'_S(x) = 0$ on $A\cap(S-der(A))$.

II. For $x\in(S-der(A))\setminus A, \overline{F}'_S(x) = -\infty$.

When $x\in(S-der(A))\setminus A$, we have $x\notin A$, but $A\cup\{x\}\in S(x)$. So

$$\begin{aligned}\overline{F}'_S(x) &= \inf_{\sigma\in S(x)}\sup\{\tfrac{F(y)-F(x)}{y-x} : y\in\sigma\,, y\neq x\}\\ &\leq \inf_{n\in\mathbf{N}}\sup\{\tfrac{F(y)-F(x)}{y-x} : y\in(A\cup\{x\})\cap[x,x+1/n),\neq x\}\\ &\leq \inf_{n\in\mathbf{N}}\sup\{\tfrac{-1-0}{y-x} : y\in(A\cup\{x\})\cap[x,x+1/n),\neq x\}\\ &\leq \inf\{(-n) : n\in\mathbf{N}\} = -\infty.\end{aligned}$$

Thus $\overline{F}'_S(x) = -\infty$.

III. For $x\in A\setminus(S-der(A)), \overline{F}'_S(x) = +\infty$.

In this case $x\in A$, and $A\cup\{x\}\notin S(x)$. Thus $A^{\sim}\cup\{x\}\in S^*(x)$. For each $m\in\mathbf{N}$ and each $\sigma\in S(x)$, $\{x\}\neq(\sigma\cap[x,x+1/m)\cap(A^{\sim}\cup\{x\}))$. Thus

$$\begin{aligned}\overline{F}'_S(x) &= \lim_{n\to\infty}\sup\{\tfrac{F(y)-F(x)}{y-x} : y\in\sigma_{x,n},\, y\neq x\}\\ &\geq \lim_{n\to\infty}\sup_{m\in\mathbf{N}}[\sup\{\tfrac{F(y)-F(x)}{y-x} : y\in\sigma_{x,n}\cap[x,x+\tfrac{1}{m})\cap(A^{\sim}\cup\{x\})\\ &\quad y\neq x\}]\\ &\geq \lim_{n\to\infty}\sup_{m\in\mathbf{N}}\{\tfrac{F(y_m)-F(x)}{y_m-x} : y_m\in\sigma_{x,n}\cap[x,x+\tfrac{1}{m})\cap A^{\sim}\}\\ &\geq \lim_{n\to\infty}\sup_{m\in\mathbf{N}}\{\tfrac{0-(-1)}{y_m-x} : y_m\in\sigma_{x,n}\cap[x,x+\tfrac{1}{m})\cap A^{\sim}\}\\ &\geq \lim_{n\to\infty}\sup\{n : n\in\mathbf{N}\} = +\infty.\end{aligned}$$

So $\overline{F}'_S(x) = +\infty$.

IV. For $x\in(A\cup(S-der(A)))^{\sim}, \overline{F}'_S(x) = 0$.

Since $x\notin A\cup(S-der(A))$, $x\notin A$ and $A\cup\{x\}\notin S(x)$. Thus $x\notin A$ and$A^{\sim}\cup\{x\}\in S^*(x)$. Hence

$$\begin{aligned}\overline{F}'_S(x) &= \limsup_{n\to\infty}\{\tfrac{F(y)-F(x)}{y-x} : y\in\sigma_{x,n}, y\neq x\}\\ &\geq \limsup_{n\to\infty}\{\tfrac{F(y)-F(x)}{y-x} : y\in\sigma_{x,n}\cap A^{\sim}\text{ and }y\neq x\}\\ &\geq \limsup_{n\to\infty}\{\tfrac{0-0}{y-x} : y\in\sigma_{x,n}\cap A^{\sim}\text{ and }y\neq x\} = 0.\end{aligned}$$

On the other hand

$$\begin{aligned}\overline{F}'_S(x) &= \inf_{\sigma\in S(x)} \sup\{\tfrac{F(y)-F(x)}{y-x} : y \in \sigma, y \neq x\} \\ &\leq \inf_{\sigma\in S(x)} \sup\{\tfrac{F(y)-0}{y-x} : y \in \sigma, y \neq x\} \leq 0,\end{aligned}$$

since $F(y) \leq 0$ and $y - x > 0$. Thus we have

$$\overline{F}'_S(x) = \begin{cases} 0 & \text{on} \quad (A \cap S - der(A)) \cup (A \cup S - der(A))^{\sim}) \\ -\infty & \text{on} \quad (S - der(A)) \setminus A \\ +\infty & \text{on} \quad \setminus(S - der(A)), \end{cases}$$

which is not a Borel measurable (measurable) function.

Note: If $\overline{F}'_S(x)$ is a Borel measurable (measurable) function, the sets $\{x : \overline{F}'_S(x) = 0\} \cap A = A \cap (S - der(A))$ and $\{x : \overline{F}'_S(x) = -\infty\} = (S - der(A)) \setminus A$

would be Borel measurable (measurable) implying the Borel measurability (measurability) of $S - der(A)$, leading to a contradiction.

The concept of continuity has been generalized in various ways. These generalizations lie between the two extreme ones, the ordinary and feeble continuity. A function F is said to be feebly continuous at a point x if there is at least one sequence of points $\{x_n\}_{n=1}^{\infty}$ with $x_n \to x$ and $F(x_n) \to F(x)$.

The function F is said to be S-discontinuous at x_0 if it fails to be S-continuous there. The set of points of S-discontinuity is contained in the set of points of ordinary discontinuity. This latter set is an F_σ set as is proved in elementary analysis.

A set A is S-negligible if and only if $A \cap (S - der(A)) = \emptyset$. Thomson in [11] shows that the set of points of S-discontinuity of an arbitrary function is a countable union of S-negligible sets.

The function F is said to have a removable S-discontinuity at x, if $S - lim_{y\to x} F(y)$ exists. Bruckner and Ceder in [5] showed that if $H \subset [a, b]$, then a necessary and sufficient condition for there to exists an everywhere discontinuous function F such that $H = \{x : \lim_{y\to x} F(y) \text{ exists }\}$, is that H be a denumerable set of type G_δ.

We use Borel measurability (measurability) of local systems to classify the Borel measurability (measurability) of non-removable points of S-discontinuity of a bounded function.

4.4 **Definition**: Let $S = \{S(x) : x \in \mathbf{R}\}$ be a local system, and F be an arbitrary bounded function. The S-oscillation of F at x, $S - \omega(F,x)$, is defined as follows:

$$S-\omega(F,x) = \inf_{\sigma \in S(x)} \sup\{F(y) : y \in \sigma,\ y \neq x\} - \sup_{\sigma \in S(x)} \inf\{F(y) : y \in \sigma,\ y \neq x\}.$$

The following lemma gives a necessary and sufficient condition on $S - \omega(F,x)$, in order for F to have a removable discontinuity at x.

4.5 **Lemma**. Let $S = \{S(x) : x \in \mathbf{R}\}$ be a local system with filtering property. Then $S - \omega(F,x) = 0$ if and only if F has a removable S-discontinuity at x.

Proof: Let F have a removable S-discontinuity at x. Then for each $\epsilon > 0$, there exists a $\sigma_\epsilon \in S(x)$ such that for all y and z in σ_ϵ different from x we have $| F(y) - F(z) | < \epsilon$. For $\epsilon = 1/n$, let $z_0 \in \sigma_{1/n}$, and $z_0 \neq x$. Then

$$\begin{aligned} S - \omega(F,x) &= \inf_{\sigma \in S(x)} \sup\{F(y) : y \in \sigma,\ y \neq x\} \\ &\quad - \sup_{\sigma \in S(x)} \inf\{F(y) : y \in \sigma,\ y \neq x\} \\ &\leq \sup\{F(y) : y \in \sigma_{1/n} y \neq x\} - \inf\{F(y) : y \in \sigma_{1/n} y \neq x\} \\ &\leq [F(z_0) + 1/n] - [F(z_0) - 1/n] \leq 2/n. \end{aligned}$$

Thus $S-\omega(F,x) \leq 2/n$ for all natural numbers $\mathbf{N}$. Hence $S-\omega(F,x) \leq 0$. On the other hand, for each natural number $\mathbf{N}$, choose $z_n \in \sigma_{1/n}, z_n \neq x$. For each $\sigma \in S(x)$ we have

$$\inf\{F(y) : y \in \sigma,\ y \neq x\} \leq \inf\{F(y) : y \in \sigma_{1/n} \cap \sigma,\ y \neq x\} \leq F(z_n) + 1/n,$$

and

$$\sup\{F(y) : y \in \sigma,\ y \neq x\} \geq sup\{F(y) : y \in \sigma_{1/n} \cap \sigma,\ y \neq x\} \geq F(z_n) - 1/n.$$

Then

$$\sup_{\sigma \in S(x)} \inf\{F(y) : y \in \sigma, \, y \neq x\} \leq F(z_n) + 1/n,$$

and

$$\inf_{\sigma \in S(x)} \sup\{F(y) : y \in \sigma, \, y \neq x\} \geq F(z_n) - 1/n.$$

Therefore $S - \omega(F, x) \geq (F(z_n) - 1/n) - (F(z_n) + 1/n) = -2/n$, for each natural number $\mathbf{N}$, which implies $S - \omega(F, x) \geq 0$.

For the converse, suppose that $S - \omega(F, x) = 0$. Thus

$$\inf_{\sigma \in S(x)} \sup\{F(y) : y \in \sigma, \, y \neq x\} = \sup_{\sigma \in S(x)} \inf\{F(y) : y \in \sigma, \, y \neq x\}.$$

For $\epsilon > 0$, there are σ_ϵ^0 and σ_ϵ^1 elements of $S(x)$ so that

$$\sup\{F(y) : y \in \sigma_\epsilon^0, \, y \neq x\} < \inf_{\sigma \in S(x)} \sup\{F(y) : y \in \sigma, \, y \neq x\} + \epsilon/2$$

and

$$\sup_{\sigma \in S(x)} \inf\{F(y) : y \in \sigma\} < \inf\{F(y) : y \in \sigma_\epsilon^1, \, y \neq x\} + \epsilon/2.$$

Let $\sigma_\epsilon = \sigma_\epsilon^0 \bigcap \sigma_\epsilon^1$. Clearly, $\sigma_\epsilon \in S(x)$, and

$$\mid \sup\{F(y) : y \in \sigma_\epsilon, \, y \neq x\} - inf\{F(y) : y \in \sigma_\epsilon, \, y \neq x\} \mid < \epsilon$$

which implies F has S-removable discontinuity at x.

4.6 **Lemma**. Let $\alpha < \Omega$ be an ordinal number, F an arbitrary bounded function, and $S = \{S(x) : x \in \mathbf{R}\}$ be a local system of the type AB_α (or of the type MB_α). Then $S - \omega(F, x)$ is a function in $B_{\alpha+1}$ (or in $B_{\alpha+2}$).

Proof: Let r be a real number, and let

$$g(x) = \inf_{\sigma \in S(x)} \sup\{F(y) : y \in \sigma, \, y \neq x\}.$$

Then

$$\{x : g(x) < r\} = \{x : \exists \sigma_r \in S(x) \text{ such that } \sup\{F(y) : y \in \sigma_r, \, y \neq x\} < r\}$$

$$= \{x : \exists r_1 < r \text{ such that } \sigma_r \subset \{x\} \bigcup \{y : F(y) < r_1\}\}$$

$$= \{x : \exists r_1 < r \text{ such that} \{x\} \bigcup \{y : F(y) < r_1\} \in S(x)\}$$

$$= \{x : \exists n \in \mathbf{N} \text{ such that} \{x\} \bigcup \{y : F(y) < r - 1/n\} \in S(x)\}$$

$$= \bigcup_{n=1}^{\infty} \{x : \{x\} \bigcup \{y : F(y) < r - 1/n\} \in S(x)\}$$

$$= \bigcup_{n=1}^{\infty} (S - der(\{y : F(y) < r - 1/n\})),$$

and therefore

$$\{x : g(x) \leq r\} = \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} (S - der(\{y : F(y) < r - 1/n + 1/m\})).$$

When the local system S is of the type AB_α (of the type MB_α), the set $\{x : g(x) < r\}$ and $\{x : g(x) > r\}$ are of the type $AB_{\alpha+1}$ (of the type $AB_{\alpha+2}$). Let

$$h(x) = \sup - \sigma \in S(x) \inf\{F(y) : y \in \sigma, \, y \neq x\}.$$

Similarly $\{x : h(x) < r\}$ and $\{x : h(x) > r\}$ are of the type $AB_{\alpha+1}$ (of the type $AB_{\alpha+2}$). Hence $S - \omega(F, x) = g(x) - h(x)$ is a function in $B_{\alpha+1}$ (in $B_{\alpha+2}$).

4.7 **Theorem.** Let $\alpha < \Omega$ be an ordinal number, $S = \{S(x) : x \in \mathbf{R}\}$ a local system of the type AB_α (of the type MB_α) with filtering property, and let F be an arbitrary bounded function. Then the set of points at which F has a non-removable S-discontinuity is of the type $AB_{\alpha+1}$ (of the type $AB_{\alpha+2}$).

Proof:From Lemma 4.6 it follows that $S - \omega(F, x)$ is a function in $B_{\alpha+1}$ (in $B_{\alpha+2}$) when the system S is of the type AB_α (of the type MB_α). Also Lemma 4.5 implies that F has a removable S-discontinuity at x if and only if $S - \omega(F, x) = 0$. Hence $\{x : x \text{is a non-removable } S\text{-discontinuity}\} = \{x : S - \omega(F, x) \neq 0\} = \{x : S - \omega(F, x) = 0\}^\sim$ is a set of the type $AB_{\alpha+1}$ (of the type $AB_{\alpha+2}$).

4.8 **Corollary.** The set of points of non-removable approximate discontinuity of a bounded function is of additive Borel class four.

Proof: Approximate continuity is S-continuity when the local system S is the density system i_{DS_1}. The set of points of non-removable approximate discontinuity is of additive Borel class four, since the local system i_{DS_1} is of the type MB_2 with filtering property.

4.9 **Remark.** Every bounded measurable function is approximately continuous almost everywhere, that is the set of points of approximate discontinuity of a bounded measurable function is of measure zero. The set of points of approximate discontinuity of an arbitrary bounded measurable function could be any set of measure zero. For example consider the function $F(x) = \chi_M(x)$ where M is any set of measure zero. The set of all points of approximate discontinuity are removable except for a set of the type AB_4.

ACKNOWLEDGMENT: The author wishes to thank Professor A.M. Bruckner for his valuable guidance.

Auther's present address: Department of mathematics, Isfahan University, Isfahan, Iran.

References

[1] A. A. Alikhani-Koopaei,*Borel measurability of extreme path derivatives*, Real Analysis Exchange, Vol. 12 No. 1 (1986-87), pp. 216-246.

[2] A. A. Alikhani-Koopaei, *Extreme path derivatives of a function with respect to a nonporous system of paths*, Real Analysis Exchange, Vol. 14 (1988-89), pp. 77-98.

[3] S. Banach, *Sur les fonctions derivees des fonctions measurables*, Fund. Math. 3 (1922), 128-132.

[4] A. M. Bruckner, *Differentiation of real functions*, Lect. Notes in Math. #659, Springer-Verlag (1978).

[5] A. M. Bruckner, J. Ceder, *A note on discontinuous functions*, Elemente Der Mathematik, Vol 28/6, 1973, pp. 147-148.

[6] A. M. Bruckner, R. J. O'Malley, B. S. Thomson, *Path derivatives: A unified view of certain genaralized derivatives*, Trans. Amer. Math. Soc. 283 (1984), 97-125.

[7] O. Hajek, *Note sur la mesurabilite B de la derivee superieue*, Fund. Math. 44 (1957), 238-240.

[8] S. Kempisty, *The extreme derivatives of functions of one or more variables*, J. London Math. Soc. 9 (1934), 303-308.

[9] L. Misik, *Extreme essential unilateral derivatives of continuous functions*, Commen. Math. 21 (1978), 235-238.

[10] L. Misik, *Extreme essential derivatives of Borel and Lebesgue measurable functions*, Math. Slovaca 29 (1979), No. 1, 25-38.

[11] B. S. Thomson, *Real functions*, Lect. Notes in Math. #1170, Springer-Verlag, (1985).

[12] B. S. Thomson, *Derivation bases on the real line*, Real Analysis Exchange, Vol. 8 (1982-83), No. 1, pp. 67-207.

[13] B. S. Thomson, *Derivation bases on the real line*, Real Analysis Exchange, Vol. 8 (1982-83), No. 2, pp. 278-442.

[14] L. Zajicek, *On cluster sets of arbitrary functions*, Fund. Math. 83 (1974), 197-217.

Received February 27, 1990