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Some properties of subclasses of Darboux functions

I would like to present some of the result of my friends Jan Jastrzębski and Tomasz Natkaniec and mine and to talk about a few classes of real functions of a real variable. They have been considered by many mathematicians, and many of us have made contributions. I start with some notation:

- $\mathcal{A}$ — the class of almost continuous functions in the sense of Stallings,
- $\mathcal{C}$ — the class of continuous functions,
- $\mathcal{C}on$ — the class of connected functions (i.e. functions with connected graphs),
- $\mathcal{C}onst$ — the class of constant functions,
- $\mathcal{D}$ — the class of Darboux functions,
- $\mathcal{F}$ — the class of functionally connected functions (i.e. $f \in \mathcal{F}$ iff for any continuous function g defined on any interval $[a, b]$ such that $f(a) < g(a)$ and $f(b) > g(b)$ — or conversely — there is a point $x_0 \in (a, b)$ for which $f(x_0) = g(x_0)$),
- lsc — the class of lower semicontinuous functions,
- usc — the class of upper semicontinuous functions,
- $\mathcal{B}_1$ — will denote the class of functions of the first class of Baire,
- $\mathcal{M}$ — the class that was considered for the first time by R. Fleissner, when he was giving the characterization of the maximal multiplicative class for $\mathcal{DB}_1$ functions;
  f belongs to the class $\mathcal{M}$ iff it is Darboux function and
  (i) if f is discontinuous at x_0 from the left, then $f(x_0) = 0$ and there is a sequence (x_n) converging to x_0 from the left such that $f(x_n) = 0$ and
  (ii) if f is discontinuous at x_0 from the right, then $f(x_0) = 0$ and there is a sequence (x_n) converging to x_0 from the right such that $f(x_n) = 0$.

If $\mathcal{X}$ is any class of functions, then we say that $\mathcal{Y}$ is maximal additive class for $\mathcal{X}$ if $f+g \in \mathcal{X}$ whenever $f \in \mathcal{X}$ and $y \in \mathcal{Y}$. This class will be denoted by $\mathcal{M}_a(\mathcal{X})$.
The similar definitions are for maximal multiplicative class — $\mathcal{M}_m(\mathcal{X})$ (i.e. $f \cdot g \in \mathcal{X}$ whenever $f \in \mathcal{X}$ and $y \in \mathcal{Y}$) and maximal laticelike class $\mathcal{M}_l(\mathcal{X})$ — $\mathcal{Y}$ is maximal laticelike class for $\mathcal{X}$ if $\max(f,g)$ and $\min(f,g)$ belong to $\mathcal{X}$ whenever $f \in \mathcal{X}$ and $y \in \mathcal{Y}$.
The problems involving the maximal additive or multiplicative classes were discussed by many authors, for example, A. M. Bruckner, R. Fleissner, J. Farkova, and many years ago T. Radakovič.
Since each function from $\mathcal{M}$ has a nowhere dense set of points of discontinuity then it is of the first class of Baire and we can state that

$$\mathcal{C} \subseteq \mathcal{M} \subseteq \mathcal{A} \subseteq \mathcal{C}on \subseteq \mathcal{F} \subseteq \mathcal{D}.$$

For functions of the first class of Baire we know (and it was written in Prof. Bruckner's monograph) that

$$\mathcal{M}_a(\mathcal{C}on) = \mathcal{M}_a(\mathcal{F}) = \mathcal{M}_a(\mathcal{DB}_1) = \mathcal{C},$$

$$\mathcal{M}_m(\mathcal{DB}_1) = \mathcal{M}, and$$

$$\mathcal{M}_a(\mathcal{D}) = \mathcal{M}_m(\mathcal{D}) = Const.$$

Moreover, similar results were obtained by J. Farkova for maximal classes with respect to maximum and minimum.
My friends and I were able to show that

$$\mathcal{M}_m(\mathcal{A}) = \mathcal{M}_m(\mathcal{C}on) = \mathcal{M}_m(\mathcal{F}) = \mathcal{M},$$

$$\mathcal{M}_l(\mathcal{A}) = \mathcal{M}_l(\mathcal{C}on) = \mathcal{M}_l(\mathcal{F}) = \mathcal{C}, and$$

$$\mathcal{M}_a(\mathcal{A}) = \mathcal{C}.$$

At the end of my talk I can formulate two problems: Is it true that:

1. $\mathcal{M}_{max}(\mathcal{A}) = \mathcal{D} \cap usc$ and $\mathcal{M}_{min}(\mathcal{A}) = \mathcal{D} \cap lsc$?
2. $\mathcal{M}_{max}(\mathcal{C}on) = \mathcal{D} \cap usc$ and $\mathcal{M}_{min}(\mathcal{C}on) = \mathcal{D}_a \cap lsc$?

where

$$\mathcal{M}_{max}(\mathcal{X}) = \{g;\ \max(f,g) \in \mathcal{X} \text{whenever} f \in \mathcal{X}\},$$

$$\mathcal{M}_{min}(\mathcal{X}) = \{g;\ \min(f,g) \in \mathcal{X} \text{whenever} f \in \mathcal{X}\}.$$