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A Restricted Symmetric Derivative for Continuous Functions of Two Variables

In this paper we are concerned with symmetric differences for real valued functions defined on the plane R^2. If $f(x,y)$ is defined on R^2, the symmetric difference at (x,y) is

$$\Delta f(x,y;h,k) = f(x+h,y+k)+f(x-h,y-k)-f(x+h,y-k)-f(x-h,y+k).$$

This difference is used to generalize second order partial derivatives as $\frac{\partial^2 f}{\partial x \partial y} = \lim_{h,k\to 0} \frac{\Delta f(x,y;h,k)}{4hk}$ if f is C^2, where $\lim_{h,k\to 0} \frac{\Delta f(x,y;h,k)}{4hk} = L$ means for every $\epsilon > 0$ there is a $\delta > 0$ so that $0 < h,k < \delta$ implies $\frac{\Delta f(x,y;h,k)}{4hk} - L < \epsilon$. In Ash, Cohen, Freiling, and Rinne [1] is the following theorem.

Theorem ACFR: If $f(x,y)$ is a continuous function on R^2 and

$$\lim_{h,k\to 0} \frac{\Delta f(x,y;h,k)}{4hk} = 0$$

for all (x,y), then there are one-variable functions a and b so that $f(x,y) = a(x) + b(y)$.

One question this led to is what happens if the ratio of h and k is controlled somehow in this limit. Specifically, fix a positive number r, and suppose we only consider differences where $k \geq rh$, which we will indicate by $\Delta_r f(x,y;h,k)$. By using certain one-dimensional partitioning properties, we will obtain the conclusion of Theorem ACFR for the restricted derivative $\lim_{k\to 0} \frac{\Delta_r f(x,y;h,k)}{4hk}$. A simple example shows that these results do not hold for arbitrary functions. Let

$$f(x,y) = \begin{cases} 1 & x < y \\ 0 & x = y \\ -1 & x > y \end{cases}.$$

Then $\Delta f(x,y;h,k) = 0$ eventually for all (x,y), yet f is not a function of x plus a function of y.

First, a few remarks and definitions that will be used in the proof. It is easy to see that $f(x,y) = a(x) + b(y)$ on some rectangular box is equivalent to all symmetric differences equal to zero in this box. Sometimes it is easier to describe situations in terms of a box and its four corners rather than use x, y, h, and k, and we will use ΔB, where B is a box with center (x,y), in place of $\Delta f(x,y;h,k)$. By *top*(B) and bottom(B) we will mean the top and bottom edges, respectively, of the box B. If B is a finite union of non-overlapping boxes $\{B_i\}_{i=1}^{n}$, then $\Delta B = \sum_{i=1}^{n} B_i$. We say that B is an r-tall box if its height is at least r times its width. For a set A, A^c is the complement of A, and int(A) is the interior of A. A full symmetric cover of an interval (a,b) is a collection, S, of subintervals of (a,b) so that for each $x \in (a,b)$ there is a $\delta(x)$ so that $0 < h < \delta(x)$ implies $[x-h, x+h] \in S$.

Theorem 1: If $f(x,y)$ is continuous and $\liminf_{k\to 0} \Delta_r f(x,y;h,k)/4hk \geq 0$ for all points (x,y), then every symmetric difference, $\Delta f(x,y;h,k)$, is non-negative.

Proof: By showing that the theorem holds for $f(x,y) + \eta xy$, where η is an arbitrary positive number, we may then let η approach zero and get the desired result, since $\frac{\partial^2 \eta xy}{\partial x \partial y} = \eta$. So, in the proof, we may assume that $\liminf_{k\to 0} \Delta_r f(x,y;h,k)/4hk \geq \eta > 0$ for all (x,y). This gives $\Delta_r f(x,y;h,k) > 0$ for k sufficiently small. Let

$$A_n = \{(x,y)\Delta_r f(x,y;h,k) \geq 0$$

for all $0 < k \leq 1/n\}$. Then R^2 is the countable union of such sets, and, since f is continuous, each A_n is closed. Applying the Baire Category Theorem, we get some A_n with nonempty interior. For any box $B \subset int(A_n)$, $\Delta B \geq 0$, since B is the finite union of r-tall boxes with height less than $1/n$. By repeating this process, we get a dense open set satisfying

(*) if B is any box contained in a component, $\Delta B \geq 0$.

Let A be the union of all open sets satisfying (*). Note that the finite union of such sets also satisfies (*). If B is any box contained in a component of A, then, by compactness, B is contained in a finite union of sets satisfying (*), so the set A also satisfies (*), and is in fact a maximal set satisfying (*). If A is all of R^2, we are done. Otherwise, A^c is a nowhere dense closed set. Since A is maximal and f is continuous, A^c is a perfect set, and each point p of A^c is a limit point of A^c minus the vertical line through p. Again, by the Baire Category Theorem, $int(A^c \cap A_n)$ is nonempty for some n, as a relative open subset of A^c. Thus, we may pick a box S with the following properties:

1) $S \cap A^c \neq \Phi$

2) if $B \subset S$ is a box contained in a component of A, then $\Delta B \geq 0$

3) if $B \subset S$ is an r-tall box with center in A^c, then $\Delta B \geq 0$.

We may assume that S is also r-tall. If $\Delta B \geq 0$ for every $B \subset S$, then $S \cap A^c = \Phi$, so we may assume that $\Delta S < 0$. We will show that the existence of such an S is a contradiction, and thus A is in fact all of R^2.

Let H be the horizontal bisector of $int(S)$. If $H \cap A^c \neq \Phi$, pick $c_1 \in H \cap A^c$ closest to the center of S, let B_1 be the largest box contained in S with center c_1, and let $V_1 = S \backslash B_1$. Since S is r-tall, so is B_1 and we have $\Delta B_1 \geq 0$. We continue the process. Suppose $B_1, ..., B_{n-1}$ have been selected with $V_i = S \backslash \cup_{k=1}^{i} B_k$, $i = 1, ..., n-1$. Pick $c_n \in H \cap V_{n-1} \cap A^c$ closest to the center of V_{n-1}, let B_n be the largest box contained in the closure of $V_n = S \backslash \cup_{k=1}^{n} B_k$. This gives a sequence of boxes, $\{B_n\}$, with $\Delta B_n \geq 0$ for each n. If the closure of the union of the $B_n's$ is all of S, then $\Delta S \geq 0$ and we are done. If not, let $T \subset S$ be the box that is the closure of S minus the union of the $B_n's$. We set $T = S$ if $H \cap A^c = \Phi$. To obtain $\Delta S \geq 0$ and our contradiction, it remains to show that $\Delta T \geq 0$. We have the fact that $H \cap int(T) \subset A$. We may assume that the two points where H crosses the boundary of T are also in A, since otherwise we simply pick concentric boxes $T' \subset T$ that are slightly less wide than T, show that $\Delta T' \geq 0$, and then use the continuity of f to get $\Delta T \geq 0$ also. Let U and L be the upper and lower halves of T. We show that $\Delta L \geq 0$, a similar argument applying to U.

We may assume that L is a rectangle of the form $[0,a] \times [0,b]$. Since $top(L) \subset A$, there is a $d \geq 0$ so that $[0,a] \times [d,b]$ is contained in a component of A. The set $Z = \{z 0 \leq z < b$ and $\Delta B \geq 0$ for all $B \subset [0,a] \times [z,b]$ with $top(B) \subset top(L)\}$ is nonempty since $d \in Z$, and we let $\beta = \inf(Z)$. If $\beta = 0$, we are done, since this says $\Delta L \geq 0$. For $\beta > 0$, we will show that $m = \max(0, 2\beta - b) \in Z$, thus contradicting the choice of β. To show $m \in Z$, we need only consider B that extend below the horizontal line $y = \beta$. It will suffice to show that $\Delta[0,a] \times [m,b] \geq 0$ since the same argument will apply to any $B \subset [0,a] \times [m,b]$ with $top(B) \subset top(L)$. Let E be the closure of the projection onto the x-axis of $A^c \cap [0,a] \times (m,\beta]$.

For each $x \in E$, define $g(x) = \sup\{y m \leq y \leq \beta$ and $(x,y) \in A^c\}$, and let $E' \subset E$ be the set on which g has upper right Dini derivate less than plus infinity or lower left Dini derivate greater than minus infinity (relative to E). Then g is of generalized bounded variation (VBG) on E'. That is, E' is a countable union of sets on which g is of bounded variation. Thus almost every (in the sense of Lebesgue measure) horizontal line hits gE' in at most countably many points (Saks [3] p 223, 279). Pick $m \leq \mu < \beta$ so that $y = \mu$ is such a horizontal line and let $\{\alpha_n\}$ be the countable level set of gE'. Since we can pick μ arbitrarily close to m, the continuity of f will give the desired result if we show that $\Delta[0,a] \times [\mu,b] \geq 0$. Since A is dense, we may assume that a portion of $y = \mu$ is contained in some component of A.

Let $\Gamma = [0,a] \times [\mu,b]$ and suppose $\Delta = \lambda < 0$. We consider four types of points in (0,a).

Type 1 If $x \in E$ and $g(x) > \mu$, consider an r-tall box B of the form $[x-h, x+h] \times [g(x)-k, g(x)+k]$ with bottom(B) contained in $y = \mu$. If $top(B) \subset [0,a] \times [\beta,b]$ then $\Delta B' \geq 0$ where $B' = [x-h, x+h] \times [\mu, b]$, since the box formed by $B' \backslash B$ is contained in $[0,a] \times [\beta,b]$. If $top(B)$ is below the line $y = \beta$, by making h sufficiently small we have $[x-h, x+h] \times [g(x)+k, \beta]$

contained in a component of A, by the definition of g and the fact that A^c is closed, and thus having a nonnegative symmetric difference. In either case we have a $\delta(x) > 0$ so that $0 < h < \delta(x)$ implies $\Delta B \geq 0$ for $B = [x-h, x+h] \times [\mu, b]$. Let $S(x)$ denote the corresponding set of intervals $[x-h, x+h]$ for x of type 1.

Type 2 If $x \in E$ and $x = \alpha_n$ for some n, by the continuity of f, we can pick $\delta(x) > 0$ so that $0 < h < \delta(x)$ implies $\Delta B \leq \lambda/2^{n+3}$ for $B = [x-h, x+h] \times [\mu, b]$. Let $S(x)$ denote the corresponding set of intervals $[x-h, x+h]$ for x of type 2.

Type 3 If $x \in E$ and $g(x) = \mu$ but $x \notin \{\alpha_n\}$, then g has $\infty and - \infty$ as upper right and lower left Dini derivates respectively. For $z < x$ let $\ell(z, n)$ be the line through the point (z, μ) with slope nr, and let $x_n' = \sup\{z < x \ell(c, n) \cap A^c \cap ([x, a] \times [\mu, \beta]) = \Phi$ for all $c < z\}$. Pick $y_n > x$ so that $(y_n, g(y_n)) \in \ell(x_n', n)$. Since $\bar{D}^+ g(x) = \infty$, we have $x_n' \neq x$ and $x_n' \uparrow x$, and if B is any nr-tall box centered at $(y_n, g(y_n))$ with bottom(B) contained in $y = \mu$, then $\Delta B \geq 0$ as long as the right edge of B does not extend beyond $x = a$. Thus we may pick n large enough so that $y_n - x < h < y_n - x_n'$ implies $\Delta B \geq 0$ for B of the form $[y_n - h, y_n + h] \times [\mu, 2g(y_n) - \mu]$. Since there are no points of A^c above B that are below the line $y = \beta$, $\Delta B' \geq 0$ for $B' = [y_n - h, y_n + h] \times [\mu, b]$. Now do the same for $z > x$ using lines of slope -nr to create corresponding intervals centered at $\tilde{y}_n < x$. Let $S(x)$ denote the collection of such intervals $[y_n - h, y_n + h]$ and $[\tilde{y}_n - h, \tilde{y}_n + h]$ for x of type 3.

Type 4 If $x \in (0, a) \backslash E$, let $\delta(x)$ be the distance from x to E. Then $0 < h < \delta(x)$ implies $\Delta B \geq 0$ for $B = [x-h, x+h] \times [\mu, \beta]$ since the interior of B is contained in a component of A. Thus $\Delta B' \geq 0$ for $B' = [x-h, x+h] \times [\mu, b]$ since $\Delta[x-h, x+h] \times [\beta, b] \geq 0$ by the definition of β. Let $S(x)$ denote the corresponding set of intervals $[x-h, x+h]$ for x of type 4.

Each point in (0,a) is then one of the four types described above. Observe that if $I_1, ..., I_n$ is a collection of nonoverlapping intervals in $\cup_{x \in I} S(x)$, then$\sum_{i=1}^{n} \Delta(I_i \times [\mu, b]) \geq \lambda/4$. Pick $t \in (0, a)$ and $\zeta > 0$ so that the portion of the line $y = \mu$ above $I = (t - \zeta, t + \zeta)$ is contained in a component of A. Every x in this interval is then type 1 or 4, so for each $x \in I$, there is a $\delta(x)$ so that $0 < h < \delta(x)$ implies $\Delta[x - h, x + h] \times [\mu, b] \geq 0$. The intervals of $\cup_{x \in I} S(x)$, that are contained in I form a full symmetric cover of I. By Lemma 3.1 in Thomson [2], for each $0 < \gamma < \zeta$ except for a set with countable closure, $[t - \gamma, t + \gamma]$ can be partitioned by elements of the full symmetric cover, and we have $\Delta[t - \gamma, t + \gamma] \times [\mu, b] \geq 0$. By the continuity of f, the above inequality is then true for all $0 < \gamma \leq \zeta$. Since we can apply this argument to any subinterval of I, $\Delta J \times [\mu, b] \geq 0$ for any $J \subset I$. In particular, $\Delta[t, t + \gamma] \times [\mu, b] \geq 0$ for all $0 < \gamma < \zeta$. Let G be the set of all $r > t$ so that, for all y in some neighborhood of r, $[t, y]$ can be partitioned by nonoverlapping intervals $I_1, .., I_n$ satisfying

(**) each $I_i \in \cup_{x \in I} S(x)$ for $i > 1$ and $I_1 \in \cup_{x \in I} S(x)$ or $\Delta I_1 \times [\mu, b] \geq 0$.

Let s be the supremum of G. We wish to show that $s = a$, which gives $\Delta[t, a] \times [\mu, \beta] \geq \lambda/4$ by the continuity of f.

Suppose $s < a$. If s is type 1, 2, or 4, then $G \cap (s - \delta(s), s)$ contains some interval J. For each $z \in J$, $[z, 2s - z] \in S(s)$ so $[t, 2s - z]$ can be partitioned by intervals satisfying (**), contradicting the choice of s. If s is type 3, there are intervals in $S(s)$ centered at some $y > s$ with left endpoints covering an interval of the form $K = (s - \epsilon, s)$. Pick an interval $J \subset G \cap K$. Then for each $z \in J$, $[z, 2y - z] \in S(s)$ so $[t, 2y - z]$ can be partitioned by intervals satisfying (**), again contradicting the choice of s.

A similar argument applies to $[0, t]$ giving $\Delta[0, t] \times [\mu, b] \geq \lambda/4$ and $\Delta = \Delta[0, a] \times [\mu, b] > \lambda$, a contradiction. Since we can pick μ arbitrarily close to m, $\Delta[0, a] \times [m, b] \geq 0$. This finishes the proof of Theorem 1.

An immediate consequence of Theorem 1 is the following.

Corollary 1: If $f(x, y)$ is continuous and $\lim_{k \to 0} \Delta_r f(x, y; h, k)/4hk = 0$ for all (x, y), then there are one-variable functions a and b so that $f(x, y) = a(x) + b(y)$.

Proof: By Theorem 1, all symmetric differences for both f *and* $-f$ are nonnegative. Thus all symmetric differences for f are equal to zero. As noted above, this is equivalent to the conclusion of the corollary.

The question remains open as to what happens if further restrictions are placed on the limit. For example, what if $rh \leq k$ is replaced by $r \leq k/h \leq R$, so that the ratio is controlled in both directions, and, as C. Freiling queried at the 14th Summer Symposium, what can be said if $h = k$?

REFERENCES

[1] J. M. Ash, J. Cohen, C. Freiling, D. Rinne, Generalizations of the wave equation, *Trans Amer Math Soc*, to appear

[2] B. Thomson, On full covering properties, *Real Analysis Exchange*, Vol. 6 No. 1 (1980-81), 77-94.

[3] S. Saks, *Theory of the integral*, Dover Pub., Inc, N.Y., 1964.

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