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ON SIMPLE CONTINUITY POINTS

Throughout this paper we assume that X and Y are topological spaces. The letters $\mathbb{N}$, $\mathbb{Q}$ and $\mathbb{R}$ stand for the set of natural, rational and real numbers, respectively.

N. Biswas in [1] introduced the following concept of simple continuity.

Definition 1. *A function $f : X \to Y$ is said to be simply continuous if for every open set V in Y the set $f^{-1}(V)$ is a union of an open set in X and a nowhere dense set in X.*

The purpose of the present paper is to introduce a suitable pointwise definition of that notion and to give a characterization of the set of all simple continuity points.

Definition 2. *We say that $f : X \to Y$ is simply continuous at a point $x \in X$ if for each open neighborhood V of $f(x)$ and for each neighborhood U of x the set $f^{-1}(V) \setminus \operatorname{int} f^{-1}(V)$ is not dense in U. Denote by N_f the set of all points at which f is simply continuous.*

REMARK 1. Let $f : X \to Y$. It is easy to verify that

(α) f is simply continuous in the sense of Biswas if and only if $N_f = X$,

(β) $Q_f \subset N_f$, where Q_f denotes the set of all points at which f is quasicontinuous (see [8]).

Lemma 1. *Let $f : X \to Y$. Then for each open set V in Y the set $N_f \cap (f^{-1}(V) \setminus \operatorname{int} f^{-1}(V))$ is nowhere dense in X.*

PROOF. Let V be an open set in Y. Put $W = f^{-1}(V) \setminus \operatorname{int} f^{-1}(V)$. It is easy to see that $W \cap \operatorname{int} \operatorname{cl} W \subset X - N_f$. Hence the set $N_f \cap W \subset (N_f \cap W) \setminus \operatorname{int} \operatorname{cl} W \subset W \setminus \operatorname{int} \operatorname{cl} W$ is nowhere dense in X.

Proposition 1. *Let $f : X \to Y$, where Y is second countable. Then the set $N_f \setminus C_f$ (where C_f is the set of all continuity points of f) is of the first category in X.*

PROOF. Let $\{B_n : n \in \mathbb{N}\}$ be a countable base of open sets in Y. Since $X \setminus C_f = \bigcup_{n=1}^{\infty} (f^{-1}(B_n) = \text{ int } f^{-1}(B_n))$, by Lemma 1 the set $N_f \setminus C_f = \bigcup_{n=1}^{\infty} (N_f \cap (f^{-1}(B_n) \setminus \text{ int } f^{-1}(B_n)))$ is of the first category in X.

The following example shows that the set $N_f \setminus C_f$ may be dense in the domain of f.

EXAMPLE 1. Let $f : \mathbf{R} \to \mathbf{R}$, $f(x) = r(x) + x$, where $r : \mathbf{R} \to \mathbf{R}$ is the Riemann function defined by

$$r(x) = \begin{cases} \frac{1}{q}, & \text{for } x = \frac{p}{q} \text{ (where } p, q \text{ are relatively prime, } q > 0), \\ 0 & \text{otherwise.} \end{cases}$$

Then $N_f \setminus C_f = \mathbf{Q}$ is dense in $\mathbf{R}$.

Definition 3. (See [8]). *Let $f : X \to Y$, where Y is a metric space with a metric d. We say that f is cliquish at a point $x \in X$ if for each $\varepsilon > 0$ and each neighborhood U of x there is a nonempty open set $G \subset U$ such that $d(f(x), f(y)) < \varepsilon$ for each $y, z \in G$. Denote by A_f the set of all points at which f is cliquish. If $A_f = X$, then f is said to be cliquish.*

REMARK 2. Let $f : X \to Y$, where Y is a metric space. Then the set $A_f \setminus N_f \subset A_f \setminus C_f$ is of the first category (see [10]). If Y is separable, then according to Proposition 1 the set $N_f \setminus A_f$ is of the first category.

The following example shows that the set $N_f \setminus A_f$ may be uncountable.

EXAMPLE 2. Let C be the Cantor discontinuum. Let $\chi : \mathbf{R} \to \mathbf{R}$ be the Dirichlet functions (i.e. $\chi(x) = 1$ for $x \in \mathbf{Q}$ and $\chi(x) = 0$ otherwise). Define $f : \mathbf{R} \to \mathbf{R}$ on the contiguous intervals (a, b) of C as follows

$$f(x) = \begin{cases} 1 + \chi(x), & \text{for } x \in (a, a + \frac{1}{3}(b - a)), \\ 2\chi(x), & \text{for } x \in (a + \frac{1}{3}(b - a), a + \frac{2}{3}(b - a)), \\ \chi(x), & \text{for } x \in (a + \frac{2}{3}(b - a), b), \end{cases}$$

and $f(x) = 0$ otherwise.

Then $N_f \setminus A_f = C \setminus \{0, 1\}$ is uncountable.

Theorem 1. Let $f : X \to Y$, where Y is a metric space with a metric d. Let at least one of the following conditions be satisfied:

(i) X is a Baire space and Y is a separable metric space,

(ii) Y is a totally bounded metric space.

Then the set $N_f \setminus A_f$ is nowhere dense in X.

PROOF. Put $G =$ int cl$(N_f \setminus A_f)$. We shall show that $G = \phi$. Suppose, by way of contradiction, that $G \neq \phi$. Put $K = G \setminus A_f$. Since the set A_f is closed (see[7]) the set K is open. We shall show that $K \neq \phi$. Since the set int $A_f \cup (X \setminus A_f)$ is dense in X and $G \cap$ int $A_f =$ int(cl$(N_f \setminus A_f) \cap A_f) \subset$ int$(A_f \setminus$ int $A_f) = \phi$, we get $\phi \neq G \cap (\text{int } A_f \cup (X \setminus A_f)) = (G \cap \text{ int } A_f) \cup K = K$.

Let $x_0 \in K$ be arbitrary. Since $x_0 \notin A_f$, there is $\varepsilon > 0$ and $L \subset K$, an open neighborhood of x_0, such that

(*) for every nonempty open set $M \subset L$ there are $y, z \in M$ such that $d(f(y), f(z)) \geq 8\varepsilon$.

We shall show that there is $v \in Y$ such that $f^{-1}(S(v, \varepsilon))$ is not nowhere dense in L (where $S(a, \eta) = \{t \in Y : d(a, t) < \eta\}$). We distinguish two cases.

a) Suppose that X is a Baire space and Y is separable. Then $Y = \bigcup_{n=1}^{\infty} S(v_n, \varepsilon)$, where $\{v_n : n \in \mathbb{N}\}$ is countable dense set in Y. Since $L = L \cap f^{-1}(\bigcup_{n=1}^{\infty} S(v_n, \varepsilon)) = \bigcup_{n=1}^{\infty} (L \cap f^{-1}(S(v_n, \varepsilon)))$, there is $k \in \mathbb{N}$ such that $L \cap f^{-1}(S(v_k, \varepsilon))$ is not nowhere dense in L.

b) Suppose that Y is totally bounded. Then there is a finite set $\{v_1, v_2, \dots, v_m\}$ in Y such that $Y = \bigcup_{n=1}^{m} S(v_n, \varepsilon)$. Since $L = L \cap f^{-1}(\bigcup_{n=1}^{m} S(v_n, \varepsilon)) = \bigcup_{n=1}^{m} (L \cap f^{-1}(S(v_n, \varepsilon)))$, there is $k \in \mathbb{N}$ such that $L \cap f^{-1}(S(v_k, \varepsilon))$ is not nowhere dense in L.

Therefore there is a nonempty open set $J \subset L$ such that $f^{-1}(S(v, \varepsilon))$ is dense in J. Put

$$D = \{y \in J : d(f(y), v) \geq 4\varepsilon\}.$$

Then in view of (*) the set D is dense in J. In the following we distinguish two cases.

α) Suppose that there is $x \in J \cap N_f$ such that $d(v, f(x)) > \varepsilon$. Put $B = \{u \in Y : d(u, v) > \varepsilon\}$. Then B is an open neighborhood of $f(x)$. Since

$f(D) \subset B$, the set $f^{-1}(B)$ is dense in J. Since $f^{-1}(S(v,\varepsilon))$ is dense in J and $f^{-1}(S(v,\varepsilon)) \cap f^{-1}(B) = \phi$, we have $\text{int}\, f^{-1}(B) \cap J = \phi$. Therefore $f^{-1}(B) - \text{int}\, f^{-1}(B)$ is dense in J, which contradicts $x \in N_f$.

β) Suppose that $d(v, f(x)) \leq \varepsilon$ for each $x \in J \cap N_f$. Since N_f is dense in J, there is $z \in J \cap N_f$. Then J is an open neighborhood of z and $S(v, 2\varepsilon)$ is an open neighborhood of $f(z)$. Put $V = \{u \in Y : d(u,v) > 2\varepsilon\}$. Since $f(D) \subset V$, the set $f^{-1}(V)$ is dense in J. Since $f^{-1}(S(v,2\varepsilon))$ is dense in J and $f^{-1}(S(v,2\varepsilon)) \cap f^{-1}(V) = \phi$, we have $\text{int}\, f^{-1}(S(v,2\varepsilon)) \cap J = \phi$. Thus $f^{-1}(S(v,2\varepsilon)) \setminus \text{int}\, f^{-1}(S(v,2\varepsilon))$ is dense in J, which contradicts $z \in N_f$.

REMARK 3. Under the assumptions of Theorem 1 every simply continuous function $f : X \to Y$ is cliquish (see [9]). Example 1 in [3] shows that those assumptions cannot be omitted.

Proposition 2. *Under the assumptions of Theorem 1 the set cl $N_f - N_f$ is of the first category in X.*

PROOF. According to Theorem 1, Remark 2 and the fact that A_f is closed (see [7]), the set $\text{cl}\, N_f \setminus N_f \subset \text{cl}((N_f \setminus A_f) \cup A_f) \setminus N_f \subset \text{cl}(N_f \setminus A_f) \cup (A_f \setminus N_f)$ is of the first category in X.

The following example shows that the assumption "Y is a metric space" in Proposition 2 cannot be omitted.

EXAMPLE 3. Let $Y = \mathbf{R}$, $\mathcal{T} = \{A \subset \mathbf{R} : \mathbf{R} \setminus A \text{ is finite or } 0 \notin A\}$. Then Y is T_4-space. Define $f : \mathbf{R} \to Y$ as follows

$$f(x) = \begin{cases} 0, & \text{for } x \in \mathbf{Q}, \\ x, & \text{otherwise.} \end{cases}$$

Then the set cl $N_f \setminus N_f$ is of the second category in $\mathbf{R}$.

We recall that a subset A of X is almost closed (see [6]) if $\text{cl int}\, A \subset A$.

Proposition 3. *Let $f : X \to Y$. Then the set N_f is almost closed.*

PROOF. Let $x \in \text{cl int}\, N_f$. Let U be an open neighborhood of x and V an open neighborhood of $f(x)$. We shall show that $f^{-1}(V) - \text{int}\, f^{-1}(V)$ is not dense in U, which yields $x \in N_f$. We distinguish two cases.

a) Suppose that there is $y \in N_f \cap U \cap f^{-1}(V)$. Since $y \in N_f$, the set $f^{-1}(V) \setminus \text{int}\, f^{-1}(V)$ is not dense in U.

b) Suppose that $f^{-1}(V) \cap U \cap N_f = \phi$. Since $x \in \text{cl int } N_f$, the set $G = U \cap \text{ int } N_f$ is nonempty open, $G \subset U$ and $f^{-1}(V) \cap G \subset f^{-1}(V) \cap U \cap N_f = \phi$. Therefore $f^{-1}(V) \setminus \text{int } f^{-1}(V)$ is not dense in U.

We recall that a topological space X is perfectly normal (see [4], p. 68) if it is normal and each closed subset of X is G_δ. A topological space is resolvable (see [2]) if it is a union of two disjoint dense sets.

Theorem 2. *Let X be a perfectly normal space such that X^d is a resolvable space (where Z^d is the set of all accumulation points of Z). Let Y be a first countable T_1-space such that $Y^d \neq \phi$. Suppose $A \subset X$ is such that*

(1) A contains all isolated points of X,

(2) A is almost closed,

(3) cl $A \setminus A$ is of the first category in X.

Then there is a function $f : X \to Y$ such that $N_f = A$.

Proof. Let $y_0 \in Y^d$. Let $\{y_n : n \in \mathbb{N}\}$ be a one-to-one sequence which converges to $y_0, y_n \neq y_0$ for all $n \in \mathbb{N}$. Since X^d is resolvable, we can write $X \setminus \text{cl } A = B \cup D$, where B and D are disjoint dense sets in $X \setminus \text{cl } A$. Since X is perfectly normal, there is a decreasing sequence $\{H_n : n \in \mathbb{N}\}$ of open sets such that $\text{cl } A = \bigcap_{n=1}^{\infty} H_n$ and $\text{cl } H_{n+1} \subset H_n$ for each $n \in N$. Put $G_0 = \phi$ and $G_n = X \setminus \text{cl } H_n$ for each $n \in \mathbb{N}$. Let $\text{cl } A \setminus A = \bigcup_{n=1}^{\infty} A_n$, where A_n are mutually disjoint and nowhere dense in X. Define a function $f : X \to Y$ as follows

$$f(x) = \begin{cases} y_0, & \text{for } x \in A \cup D, \\ y_n, & \text{for } x \in A_n \cup ((G_n \setminus G_{n-1}) \cap B). \end{cases}$$

We shall show that $N_f = A$. We distinguish four cases.

I) Suppose that $x_0 \in A$. Then $f(x_0) = y_0$. Let U be an open neighborhood of x_0 and V an open neighborhood of $f(x_0)$. Then there is $k \in \mathbb{N}$ such that $y_n \in V$ for each $n > k$. Put $G = H_k \cap U$. Then G is an open neighborhood of x_0 and $G \subset U$. Since $G \cap G_n = \phi$ for each $n \leq k$, we have $G \setminus \bigcup_{n=1}^{k} A_n \subset G \cap f^{-1}(V)$. Since A_n are nowhere dense sets, we have $\text{int}(G - \bigcup_{n=1}^{k} A_n) \neq \phi$. Hence $\phi \neq \text{int}(G \cap f^{-1}(V)) = G \cap \text{ int } f^{-1}(V)$. Therefore $f^{-1}(V) \setminus \text{int } f^{-1}(V)$ is not dense in U. Thus $x_0 \in N_f$.

II) Suppose that $x_o \in (G_k \setminus G_{k-1}) \cap B$ for some $k \in \mathbb{N}$. Put $U = X - \text{ cl } A$ and $V = Y \setminus \{y_0\}$. Then U is an open neighborhood of x_0 and V is an open neighborhood of $f(x_0) = y_k$. We have $f^{-1}(V) \cap U = B$. Since B is dense in U and $\text{int } B = \phi$, the set $f^{-1}(V) - \text{ int } f^{-1}(V)$ is dense in U. Thus $x_0 \notin N_f$.

III) Suppose that $x_0 \in D$. Since $x_0 \in X \setminus \text{cl } A$, there is $k \in \mathbb{N}$ such that $x_0 \in G_k \setminus G_{k-1}$. Put $U = G_k$ and $V = Y \setminus \{y_1, y_2, \dots, y_k\}$. Then U is an open neighborhood of x_0 and V is an open neighborhood of $f(x_0) = y_0$. Since $D \subset f^{-1}(V)$, the set $f^{-1}(V)$ is dense in U. Since $U \cap B$ is dense in U and $U \cap B \cap f^{-1}(V) = \phi$, we have $U \cap$ int $f^{-1}(V) = \phi$. Hence $f^{-1}(V) \setminus \text{int } f^{-1}(V)$ is dense in U. Thus $x_0 \notin N_f$.

IV) Suppose that $x_0 \in A_k$ for some $k \in \mathbb{N}$. Put $U = X \setminus \text{cl int } A$ and $V = Y \setminus \{y_0\}$. Since the set A is almost closed, we have $x_0 \in A_k \subset X \setminus A \subset U$. Therefore U is an open neighborhood of x_0 and V is an open neighborhood of $f(x_0) = y_k$. Since $f^{-1}(V) = B \cup (\text{cl } A \setminus A)$, int $f^{-1}(V) = \phi$ and cl $f^{-1}(V) = (X \setminus \text{cl } A) \cup (\text{cl } A - \text{ int } A) = \text{cl}(X \setminus A)$. So $U = X \setminus \text{cl int } A \subset \text{cl}(X \setminus A) = \text{cl}(f^{-1}(V) \setminus \text{int } f^{-1}(V))$. Thus $x_0 \notin N_f$.

Theorem 3. *Let X be a perfectly normal space such that X^d is a resolvable space. Let Y be a metric space such that $Y^d \neq \phi$. Let us assume that (i) or (ii) is satisfied. Let $A \subset X$. Then there is a function $f : X \to Y$ such that $N_f = A$ if and only if the set A has the properties (1), (2) and (3).*

REMARK 4. Theorems 1 and 3 are true if instead of (i) or (ii) we require

(iii) X is a k-Baire space (see [5]) and Y is a metric space with weight (see [4, p. 27]) less than k.

REMARK 5. It was shown in [7] that a set A is Q_f for some f if and only if int cl $A \setminus A$ is first category, which is stronger than condition (3). Whereas the sets A_f are generally closed, and the sets C_f are generally G_δ sets, the sets Q_f and N_f don't even have to be Lebesque measurable. However, they must have the Baire property.

Theorem 4. *Let $f : X \to Y$, where X is a Baire space and Y is a separable metric space. Then the following three statements are equivalent:*

(u) $X \setminus N_f$ is a set of the first category in X,

(v) N_f is a dense set in X,

(w) f is cliquish.

PROOF. *(u) $\Rightarrow$ (v): Obvious.*

(v) ⇒ (w): We have $X \setminus A_f \subset (X \setminus N_f) \cup (N_f \setminus A_f) = (cl\, N_f \setminus N_f) \cup (N_f \setminus A_f)$. *Therefore according to Theorem 1 and Proposition 2* $X \setminus A_f$ *is an open set of the first category and hence* $X \setminus A_f = \phi$.
(w) ⇒ (u): Follow's from Remark 2.

The Riemann function shows that the assumption (v) in Theorem 4 cannot be replaced by the assumption "$N_f = X$*".*

References

[1] Biswas,N., *On some mappings in topological spaces*, Calcutta Math. Soc., 61 (1969) 127–135.

[2] Bolstein, R., *Sets of points of discontinuity*, Proc. Amer. Math. Soc. 38 (1973), 193–197.

[3] Doboš, J., *Simple continuity and cliquishness*, Časopis pěst. mat. 112 (1987), 355–358.

[4] Engelking, R., *General Topology*, PWN, Warszawa, 1977.

[5] Haworth, R.C. and McCoy, R.A., *Baire spaces*, Dissertationes Math. 141, Warszawa, 1977.

[6] Holá, Ľ., *A remark on almost continuous multifunctions*, Math. Slovaca, 38 (1988), 325–331.

[7] Lipinski, J.S. and Šalát, T., *On the points of quasicontinuity and cliquishness of functions*, Czechoslovak Math. J. 21 (1971), 484–489.

[8] Marcus, S.,*Sur les fonctions quasicontinues au sens de S. Kempisty*, Colloq. Math. 8 (1961), 47–53.

[9] Neubrunnová, A., *On certain generalizations of the notion of continuity*, Mat. časopis, 23 (1973), 374–380.

[10] Neubrunnová, A., *On quasicontinuous and cliquish functions*, Časopis pěst. mat. 99 (1974), 109–114.

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