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INTEGRATION BY PARTS FOR THE PERRON INTEGRAL

1. INTRODUCTION. For a definition of the Perron integral the reader is referred to Saks [3] and for a simple proof of, and detailed history of, the integration by parts formula, to Bullen, [1] and [2], respectively.

The proof presented here is based on the following necessary and sufficient conditions for Perron integrability as given by Skljarenko [4]:

Theorem A (Skljarenko). *A finite function f is Perron integrable on $[a,b]$ with indefinite integral F if and only if corresponding to $\varepsilon > 0$, there exist a function $\delta(x) > 0$ and a non-decreasing absolutely continuous (AC) function $m(x)$ on $[a,b]$ with $m(a) = 0$ and $m(b) < \varepsilon$, such that*

$$|F(x+h) - F(x) - hf(x)| < |m(x+h) - m(x)|$$

for $x, x+h \in [a,b]$ and $0 < |h| < \delta(x)$.

2. INTEGRATION BY PARTS. We prove the following formulation of integration by parts:

Theorem 1. *Suppose f is finite valued and Perron integrable on $[a,b]$. Let*

$$F(x) = \int_a^x f(t)dt, \quad G(x) = \int_a^x g(t)dt$$

where g is of bounded variation. Then fG is Perron integrable on $[a,b]$ and

$$\int_a^b f(t)G(t)dt = F(b)G(b) - \int_a^b F(t)g(t)dt$$

Proof: We may assume without loss of generality that g is monotonic increasing and $g(a) = 0$. We note also that Fg is Lebesgue integrable and hence Perron integrable. We can write

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\begin{aligned}
&\left|\int_x^{x+h} F(t)g(t)dt + hf(x)G(x) - F(x+h)G(x+h) + F(x)G(x)\right| \\
&\quad \le \left|\int_x^{x+h} F(t)g(t)dt - hF(x)g(x)\right| \\
&\qquad + |F(x)(-G(x+h) + G(x) + hg(x))| \\
&\qquad + |G(x)(-F(x+h) + F(x) + hf(x))| \\
&\qquad + |(F(x) - F(x+h))(G(x+h) - G(x))| \\
&\quad \equiv K_1 + K_2 + K_3 + K_4
\end{aligned}
$$

Since Fg is Perron integrable there exist (by Theorem A) a function $\delta_1(x) > 0$ and a non-decreasing AC function $m_1(x)$ on $[a,b]$ with $m_1(a) = 0$ and $m_1(b) < \varepsilon/4$ such that $K_1 < |m_1(x+h) - m_1(x)|$ for $x, x+h \in [a,b]$ and $0 < |h| < \delta_1(x)$.

Let $M = \max_{x\in[a,b]}\{|F(x)|, |G(x)|\}$. Then $K_2 \le M|G(x+h) - G(x) - hg(x)|$ and by Theorem A again there exist $\delta_2(x) > 0$ and a non-decreasing AC function $m_2(x)$ with $m_2(a) = 0$ and $m_2(b) < \frac{\varepsilon}{4M}$ such that writing $m_3(x) = Mm_2(x)$ we have

$$K_2 < M|m_2(x+h) - m_2(x)| = |m_3(x+h) - m_3(x)|$$

for $x, x+h \in [a,b]$ and $0 < |h| < \delta_2(x)$.

We may treat K_3 similarly and so there exist $\delta_3(x) > 0$ and a non-decreasing AC function $m_4(x)$ with $m_4(a) = 0$ and $m_4(b) < \varepsilon/4$ such that

$$K_3 < |m_4(x+h) - m_4(x)|$$

for $x, x+h \in [a,b]$ and $0 < |h| < \delta_3(x)$.

Finally,

$$K_4 \le |F(x+h) - F(x)|\,|G(x+h) - G(x)|$$

where, from its definition, $G(x)$ is a monotonic increasing AC function with $G(a) = 0$. Since $F(x)$ is continuous, there exists $\delta_4 > 0$ so that

$$|F(x+h) - F(x)| < \frac{\varepsilon}{4M}, \text{ for } 0 < |h| < \delta_4.$$

Defining $m_5(x) = \frac{\varepsilon G(x)}{4M}$, we have $m_5(a) = 0, m_5(b) < \varepsilon/4$ and

$$K_4 \le |m_5(x+h) - m_5(x)|$$

for $x, x+h \in [a,b]$ and $0 < |h| < \delta_4$.

Now writing $m(x) = m_1(x) + m_3(x) + m_4(x) + m_5(x)$ and $\delta(x) = \min(\delta_1(x), \delta_2(x), \delta_3(x), \delta_4)$, the proof of Theorem 1 follows from Theorem A.

References

[1] Bullen, P.S., *A simple proof of integration by parts for the Perron integral*, Canad. Math. Bull. 28 (2) (1985), 195-199.

[2] Bullen, P.S., *A survey of integration by parts for Perron integrals*, J. Australian Math. Soc. Ser. A 40 (1986), 343-363.

[3] Saks, S. *Theory of the integral* (Hafner, New York, 1939).

[4] Skljarenko, V.A., *On integration by parts in Burkill's* SCP-*integral*, Math. Sbornik 40 (1981), 567-582.

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