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Globally Small Riemann Sums and the Henstock Integral

1 Introduction

The concept of GSRG (globally small Riemann sums) was first defined by Lu [3; p.114] in an attempt to characterize the Henstock integral. We recall that a measurable function f defined on $[a,b]$ is said to have GSRS if for every $\varepsilon > 0$ there exists a positive integer N such that for every $n \geq N$ there is $\delta_n(\xi) > 0$ and for every δ_n-fine division $D = \{([u,v],\xi)\}$ of $[a,b]$ we have

$$\Big| \sum_{|f(\xi)|>n} f(\xi)(v-u)\Big| < \varepsilon$$

where the sum is taken over D for which $|f(\xi)| > n$. For the notation and the definition of δ-fine divisions, see [3; p.5].

Then we can prove [3; p.115].

<u>**Theorem 1.**</u> A measurable function f has GSRS if and only if f is Henstock integrable to $F(a,b)$ on $[a,b]$ and $F_n(a,b) \to F(a,b)$ as $n \to \infty$ where $F_n(a,b)$ denotes the integral of f_n on $[a,b]$ with $f_n(x) = f(x)$ when $|f(x)| \leq n$ and 0 otherwise.

There is an example [3; p.115] to show that a Henstock integrable function does not necessarily have GSRS. Also, every measurable function which is Lebesgue integrable has GSRS though not conversely.

In this paper, we modify GSRS and achieve the characterization of the Henstock integral using modified GSRS.

The following lemma due to Lu will be useful later.

<u>**Lemma 2.**</u> If f is Henstock integrable on $[a,b]$ then there is a sequence $\{X_n\}$ of closed subsets of $[a,b]$ such that $X_n \subset X_{n+1}$ for all n, $[a,b] \setminus \cup_{n=1}^{\infty} X_n$ is of measure

zero, f is absolutely Henstock integrable on each X_n and

$$\lim_{n\to\infty} \int_{X_n} f(x)dx = \int_a^b f(x)dx.$$

Proof. The proof resembles that of showing a conditionally convergent series can be rearranged to converge to any real number. Consider

$$a_n = \int_{A_n} f(x)dx \text{ and } b_n = \int_{B_n} f(x)dx$$

where $A_n = \{x;\ n-1 \le f(x) < n\}$ and $B_n = \{x;\ -n \le f(x) < -n+1\}$ for $n = 1, 2, \ldots$. Let I denote the integral of f on $[a,b]$. Obviously, I is the sum of a_n's and b_n's in some order. Four cases may occur, namely, (i) $\sum a_n < \infty$, $\sum b_n > -\infty$; (ii) $\sum a_n = \infty$, $\sum b_n > -\infty$; (iii) $\sum a_n < \infty$, $\sum b_n = -\infty$; and (iv) $\sum a_n = \infty$, $\sum b_n = -\infty$. In the first case, f is Lebesgue integrable and the result follows directly. In the second case, put $f_1(x) = f(x)$ when $x \in \bigcup_{n=1}^{\infty} A_n$ and 0 elsewhere, and $f_2(x) = f(x)$ when $x \in \bigcup_{n=1}^{\infty} B_n$ and 0 elsewhere. Then f_2 is Lebesgue integrable and hence Henstock integrable. It follows that $f_1 = f - f_2$ is also Henstock integrable and indeed it is Lebesgue integrable. But this is impossible for $\sum a_n = -\infty$. Similarly the third case does not occur. It remains to check the fourth case.

First, let $I > 0$ and construct the following two sequences of positive integers. Define $n(1)$ so that

$$\sum_{i=1}^{n(1)-1} a_i \le I < \sum_{i=1}^{n(1)} a_i.$$

Next, define $m(1)$ and $n(2)$ so that

$$\sum_{i=1}^{n(1)} a_i + \sum_{i=1}^{m(1)} b_i < I \le \sum_{i=1}^{n(1)} a_i + \sum_{i=1}^{m(1)-1} b_i,$$

$$\sum_{i=1}^{n(2)-1} a_i + \sum_{i=1}^{m(1)} b_i \le I < \sum_{i=1}^{n(2)} a_i + \sum_{i=1}^{m(1)} b_i.$$

Finally, define $m(k)$ and $n(k+1)$ inductively for $k = 2, 3, \ldots$, so that

$$\sum_{i=1}^{n(k)} a_i + \sum_{i=1}^{m(k)} b_i < I \le \sum_{i=1}^{n(k)} a_i + \sum_{i=1}^{m(k)-1} b_i,$$

$$\sum_{i=1}^{n(k+1)-1} a_i + \sum_{i=1}^{m(k)} b_i \le I < \sum_{i=1}^{n(k+1)} a_i + \sum_{i=1}^{m(k)} b_i.$$

We shall use the fact that if

$$\int_A f < I < \int_B f \text{ or } \int_A f > I > \int_B f$$

where A and B are measurable sets with $A \subset B$, then there exists a measurable set X such that

$$A \subset X \subset B \text{ and } \int_X f = I.$$

Therefore, in view of the above inequalities, we may choose a measurable set X_k so that

$$\left(\cup_{i=1}^{n(k)} A_i\right) \cup \left(\cup_{i=1}^{m(k)} B_i\right) \supset X_{2k-1} \supset \left(\cup_{i=1}^{n(k)} A_i\right) \cup \left(\cup_{i=1}^{m(k)-1} B_i\right)$$

$$\left(\cup_{i=1}^{n(k+1)-1} A_i\right) \cup \left(\cup_{i=1}^{m(k)} B_i\right) \subset X_{2k} \subset \left(\cup_{i=1}^{n(k+1)} A_i\right) \cup \left(\cup_{i=1}^{m(k)} B_i\right)$$

when $k = 1, 2, 3, \dots$ and for each k

$$\int_{X_k} f(x)dx = I.$$

If X_k is not closed, we may choose a closed set $Y_k \subset X_k$ so that

$$|\int_{Y_k} f(x)dx - I| < 2^{-k}.$$

We can verify that all the conditions in Lemma 2 are satisfied with X_k replaced by suitable Y_k if necessary. The case when $I \leq 0$ is similar.

We remark that in Lemma 2 we may choose X_n so that the primitive F of f is $AC^*(X_n)$ for each n. For definition of $AC^*(X)$, see [3; p.27]. The proof is the same as that of Lemma 2 except that we put

$$A_n = \{x; |g_{n-1}(x)| \leq f(x) < |g_n(x)|\},$$

$$B_n = \{x; -|g_n(x)| \leq f(x) < -|g_{n-1}(x)|\},$$

where $[a,b] = \cup_{n=1}^{\infty} Z_n$, F is $AC^*(Z_n)$ for each n, $g_0(x) = 0$ for all x, and $g_n(x) = f(x)$ when $x \in Z_n$, zero otherwise.

2 Modified GSRS

A measurable function f defined on $[a,b]$ is said to have FSRS if for every $\varepsilon > 0$ there exist a nonnegative function g which is Lebesgue integrable on $[a,b]$ and $\delta(\xi) > 0$ such that for every δ-fine division D of $[a,b]$ we have

$$|\sum_{|f(\xi)|>g(\xi)} f(\xi)(v-u)| < \varepsilon$$

where the sum is taken over D which $|f(\xi)| > g(\xi)$. Here FSRS stands for functionally small Riemann sums. Obviously, FSRS includes GSRS as a special case.

Theorem 3. A measurable function f has FSRS if and only if f is Henstock integrable on $[a,b]$.

Proof. Suppose f has FSRS. Given $\varepsilon > 0$ there are Lebesgue integrable function g and $\delta(\xi) > 0$ such that the corresponding Riemann sums are small. Put

$$\begin{aligned} f^*(x) &= f(x) \quad \text{when} \quad |f(x)| \le g(x), \\ &= 0 \qquad \text{when} \quad |f(x)| > g(x). \end{aligned}$$

Then f^* being measurable and dominated is also Lebesgue integrable on $[a,b]$. We may assume that for the same $\delta(\xi) > 0$ and any two δ-fine divisions D_1 and D_2 of $[a,b]$ we have

$$|(D_1)\sum f^*(\xi)(v-u) - (D_2)\sum f^*(\xi)(v-u)| < \varepsilon.$$

Consequently, using the FSRS property and the above inequality we have

$$|(D_1)\sum f(\xi)(v-u) - (D_2)\sum f(\xi)(v-u)| < 3\varepsilon.$$

Hence f is Henstock integrable on $[a,b]$.

Conversely, if f is Henstock integrable on $[a,b]$ then by Lemma 2 given $\varepsilon > 0$ there is a measurable subset $X \subset [a,b]$ such that f is absolutely Henstock integrable on X and

$$|\int_X f(x)dx - I| < \varepsilon$$

where I denotes the integral of f on $[a,b]$. Also, given $\varepsilon > 0$ there is $\delta(\xi) > 0$ such that for any δ-fine division D of $[a,b]$ we have

$$|(D)\sum f(\xi)(v-u) - I| < \varepsilon.$$

Put $g(x) = f(x)$ when $x \in X$ and zero otherwise. Then g is Lebesgue integrable on $[a,b]$. We may assume for the same $\delta(\xi) > 0$ and for any δ-fine division D of $[a,b]$ we have

$$|(D)\sum g(\xi)(v-u) - \int_a^b g(x)dx| < \varepsilon.$$

Consequently, using the above three inequalities we obtain

$$\begin{aligned} |\sum_{|f(\xi)|>|g(\xi)|} f(\xi)(v-u)| &= |(D)\sum f(\xi)(v-u) - (D)\sum g(\xi)(v-u)| \\ &< 3\varepsilon. \end{aligned}$$

That is, f has FSRS.

We remark that we cannot replace g by a constant in the proof of Theorem 3. That is why GSRS and FSRS are not equivalent. Further, in the proof we may choose X so that $|[a,b]-X|$ is arbitrarily small. Also, as seen in the proof, in the difinition of FSRS instead of g we may say: for every $\varepsilon > 0$ there exists a closed set X on which f is Lebesgue integrable and $\delta(\xi) > 0$ such that for every δ-fine division K of $[a,b]$ we have

$$|\sum_{\xi \notin X} f(\xi)(v-u)| < \varepsilon$$

where the sum is taken over D for which $\xi \notin X$.

In what follows, we give another version of modified GSRS. Ding [1] gives a Riemann-type definition to the Lebesgue integral, called the RL integral, using a constant δ and a small open set G. A countable extension of the RL integral gives rise to the DL integral [2], which is equivalent to the Kunugi integral and not the Henstock integral. Here we give a version which is equivalent to the Henstock integral.

Theorem 4. A function f is Henstock integrable on $[a,b]$ if and only if there is a number A such that for every $\varepsilon > 0$ and $\eta > 0$ there exist a constant $\delta_0 > 0$ and an open set G with $|G| < \eta$ such that for any division $D = \{([u,v],\xi)\}$ of $[a,b]$ with $0 < v-u < \delta_0$ and $\xi \in [u,v]-G$ we have

$$|\sum_{\xi \notin G} f(\xi)(v-u) - A| < \in$$

and there exists $\delta(\xi) > 0$ satisfying $(\xi - \delta(\xi),\ \xi + \delta(\xi)) \subset G$ when $\xi \in G$ such that for any δ-fine division D of $[a,b]$ we have

$$|\sum_{\xi \in G} f(\xi)(v-u)| < \in .$$

Proof. Suppose f is Henstock integrable on $[a,b]$. Given $\varepsilon > 0$ there is $\delta(\xi) > 0$ such that for any δ-fine division D of $[a,b]$ we have

$$|(D)\sum f(\xi)(v-u) - A| < \in .$$

Given $\eta > 0$ there are $\delta_0 > 0$ and an open set G_1 such that $|G_1| < \eta/2$ and

$$G_1 \supset \{x;\ 0 < \delta(x) < \delta_0\}.$$

This is possible since we may assume $\delta(x)$ to be measurable [3; p.60]. Next, since f has FSRS by Theorem 3, we can choose g bounded by N and $\delta(\xi) > 0$ satisfying the condition of FSRS. In view of the remarks at the end of the proof of Theorem 3, we can choose g bounded by N, $G = \{x;\ |f(x)| > g(x)\}$ open since $\{x;\ |f(x)| \leq g(x)\}$ can be made closed, and further $|G_2| < \eta/2$. We may choose $\delta(\xi)$ here to be the same as above.

Put $G = G_1 \cup G_2$. Then $|G| < \eta$. Define $\delta^*(\xi) = \delta_0$ when $\xi \notin G$ and $(\xi - \delta^*(\xi),\ \xi + \delta^*(\xi)) \subset (\xi - \delta(\xi),\ \xi + \delta(\xi)) \cap G$ when $\xi \in G$. For any δ^*-fine division D of $[a, b]$ we have

$$\begin{aligned} \sum_{\xi \in G} |f(\xi)(v-u)| &\leq |\sum_{\xi \in G_2} f(\xi)(v-u)| + |\sum_{\xi \in G - G_2} f(\xi)(v-u)| \\ &< \in + N\eta. \end{aligned}$$

and consequently,

$$\begin{aligned} |\sum_{\xi \notin G} f(\xi)(v-u) - A| &\leq |(D)\sum f(\xi)(v-u) - A| + |\sum_{\xi \in G} f(\xi)(v-u)| \\ &< 2\varepsilon + N\eta \end{aligned}$$

Hence all the conditions are satisfied.

Conversely, suppose the conditions are satisfied. We put $\delta^*(\xi) = \delta_0$ when $\xi \notin G$ and $\delta^*(\xi) = \delta(\xi)$ when $\xi \in G$. Then f is Henstock integrable on $[a, b]$ using $\delta^*(\xi)$.

3 Convergence theorems

To illustrate the use of FSRS, we shall prove in the following a convergence theorem.

Theorem 5. Let $\{f_n\}$ be a sequence of measurable functions such that $f_n(x) \to f(x)$ almost everywhere in $[a, b]$ as $n \to \infty$. If f_n have FSRS uniformly in n, then f is Henstock integrable on $[a, b]$ and we have

$$\lim_{n \to \infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx.$$

Proof. By assumption, for every $\varepsilon > 0$ there exist a Lebesgue integrable function g and $\delta(\xi) > 0$ such that for every δ-fine division D of $[a, b]$ and for all n we have

$$|\sum_{|f_n(\xi)| > g(\xi)} f_n(\xi)(v-u)| < \varepsilon,$$

where g and $\delta(\xi)$ are independent of n. Hence we may replace f_n by f and therefore f has FSRS and is Henstock integrable on $[a, b]$. Put $f_n^*(x) = f_n(x)$ when $|f_n(x)| \leq$

$g(x)$ and 0 otherwise. Also, $f^*(x) = f(x)$ when $|f(x)| \leq g(x)$ and 0 otherwise. Since f_n and f_n^* are all Henstock interable on $[a,b]$, there is $\delta_n(x) > 0$ with $\delta_n(\xi) \leq \delta(\xi)$ such that for any δ_n-fine division D we have

$$|(D)\sum f_n^*(\xi)(v-u) - \int_a^b f_n^*(x)dx| < \varepsilon.$$

$$|(D)\sum f_n(\xi)(v-u) - \int_a^b f_n(x)dx| < \varepsilon.$$

Combining the three inequalities, we have

$$|\int_a^b f_n^*(x)dx - \int_a^b f_n(x)dx| < 3\varepsilon.$$

Also, the above holds with f_n^* and f_n replaced by f^* and f.

It follows from Lebesgue's dominated convergence theorem that

$$\lim_{n\to\infty} \int_a^b f_n^*(x)dx = \int_a^b f^*(x)dx.$$

Thus for sufficiently large n we have

$$|\int_a^b f_n(x)dx - \int_a^b f(x)dx| < 7\varepsilon.$$

Hence the proof is complete.

Corollary 6. Let $f_n(x) \to f(x)$ almost everywhere in $[a,b]$ as $n \to \infty$. If $g(x) \leq f_n(x) \leq h(x)$ for almost all x in $[a,b]$ and all n where all functions are measurable and have FSRS, then f is Henstock integrable on $[a,b]$ and we have

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx.$$

Proof. For every $\varepsilon > 0$ there exist X_1 on which h is Lebesgue integrable and $\delta_1(\xi) > 0$ such that for every δ_1-fine division of $[a,b]$ we have

$$|\sum_{\xi \notin X_1} h(\xi)(v-u)| < \varepsilon.$$

Also, there exist X_2 on which g is Lebesgue integrable and $\delta_2(\xi) > 0$ such that for every δ_2-fine division of $[a,b]$ we have

$$|\sum_{\xi \notin X_2} g(\xi)(v-u)| < \varepsilon.$$

Suppose the above holds with $\delta_1(\xi), \delta_2(\xi)$ replaced by $\delta(\xi) = \min\{\delta_1(\xi), \delta_2(\xi)\}$ and X_1, X_2 by X. Then for any δ-fine division of $[a, b]$ and for all n we have

$$\begin{aligned} |\sum_{\xi \notin X} f_n(\xi)(v-u)| &\le |\sum_{\xi \notin X} h(\xi)(v-u)| \\ &\quad + |\sum_{\xi \notin X} g(\xi)(v-u)| \\ &< 2\varepsilon. \end{aligned}$$

Hence f_n have FSRS uniformly in n and by Theorem 5 the result follows.

It is easy to see that we may replace $\delta_1(\xi), \delta_2(\xi)$ by $\delta(\xi)$. It remains to show that we may replace X_1, X_2 by X. Since $h - g$ is Henstock integrable on $[a, b]$ and nonnegative, its primitive is absolutely continuous [3; p.18]. Modify $\delta(\xi)$ if necessary and there exists $\eta > 0$ such that for $Y \subset [a, b]$ with $|Y| < \eta$ and for any δ-fine division of $[a, b]$ we have

$$|\sum_{\xi \in Y} \{h(\xi) - g(\xi)\}(v-u)| < 2\varepsilon.$$

Note that we may choose X_1 above so that $|[a, b] - X_1| < \eta$. Then for any δ-fine division of $[a, b]$ we have

$$\begin{aligned} |\sum_{\xi \notin X_1} g(\xi)(v-u)| &\le |\sum_{\xi \notin X_1} \{h(\xi) - g(\xi)\}(v-u)| + |\sum_{\xi \notin X_1} h(\xi)(v-u)| \\ &< 3\varepsilon. \end{aligned}$$

Consequently, we may choose $X = X_1$, i.e. we may replace X_2 above by X_1 with the corresponding Riemann sum less than 3ε and the proof is complete.

Corollary 6 can be proved easily by writing $0 \le f_n - g \le h - g$ and the use of the Lebesgue's dominated convergence theorem. It does not seem possible to deduce the generalized dominated convergence theorem [4] as a consequence of Theorem 5.

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