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ON COMPLETENESS

Dedicated to John C. Oxtoby

When introducing the completion of a metric space, Hausdorff stated that this notion is analogous to Dedekind's completion of the rational numbers. At first glance, the analogy here appears to be somewhat superficial, since the usual constructions utilizing Cauchy sequences and Dedekind cuts, respectively, exhibit no readily discernible similarities. In this article we shall give depth to Hausdorff's statement by revealing an analogous construction underlying these notions.

Let $\mathcal{C}$ be a family of subsets of a set of points X. The nonempty sets in $\mathcal{C}$ are called regions. A subset of a region A which is itself a region is called a subregion of A.

Definition. Let $\Psi = \langle \psi_n : n \in \mathbb{N} \rangle$ be a sequence of mappings from $\mathcal{C}$ to $\mathcal{C}$ having the property that, for every region A and every $n \in \mathbb{N}$, $\psi_n(A)$ is a subregion of A. (In the case that $\emptyset \in \mathcal{C}$, we take $\psi_n(\emptyset) = \emptyset$ for every n.) Then the triple $\mathfrak{X} = \langle X, \mathcal{C}, \Psi \rangle$ is called a basic system.

When defining $\psi_n(A)$ for regions A in examples below we shall assume a given well-ordering of $\mathcal{C}$.

Example A. Let (X, d) be a metric space and let $\mathcal{C}$ be the family of all open sets in X. For each region A and each $n \in \mathbb{N}$ we define $\psi_n(A) = A$ whenever $\operatorname{diam}(A) \leq \frac{1}{n}$; otherwise, $\psi_n(A)$ is defined to be the first subregion B of A (relative to the assumed well-ordering of $\mathcal{C}$) with $\operatorname{diam}(B) \leq \frac{1}{n}$.

We note that completeness (in the sense of Cauchy) of a metric (or pseudometric) space X has the following equivalent characterization (cf. [7]):

(m) X is complete if and only if every descending sequence of closed spheres in X whose diameters converge to 0 has a nonempty intersection.

Example B. Let $(Y, <)$ be an ordered set having no smallest or largest elements and containing a denumerable everywhere dense set $Q = \{r_n : n \in \mathbb{N}\}$ and let

$\{I_m : m \in \mathbb{N}\}$ be an enumeration of all open intervals of Y with endpoints in Q. Let X be a subset of Y and let $\mathcal{C} = \{C_m : m \in \mathbb{N}\}$, where $C_m = I_m \cap X$. For each $n \in \mathbb{N}$ and each region C_m, define $\psi_n(C_m) = C_k$, where k is the smallest index greater than or equal to m such that C_k is a subregion of C_m and I_k is a subinterval of I_m which contains at most one of the elements $r_1, r_2, \ldots, r_{n+1}$ and none of these elements is an endpoint of I_k.

A subset C of an ordered set Z is called a generalized interval in Z if it satisfies the condition: For all elements $x, y \in C$, $z \in Z$, if $x \leq z \leq y$ then $z \in C$. Note that if I is a generalized interval in an ordered set Y and X is a subset of Y then $I \cap X$ is a generalized interval in X; in particular, the sets C_m above are generalized intervals in X.

We note that completeness (in the sense of Dedekind) of the ordered set X above has the equivalent characterization

(o) X is complete if and only if every descending sequence of bounded, closed intervals in X has a nonempty intersection.

The analogous characterizations (m) and (o) suggest that a general notion of completeness, based on descending sequences of sets and nonempty intersections, will effect a unification of the aforementioned metric and order completion results. It is of some interest to note that our approach to the unification of metrical and order-theoretic analogies and our approach to the unification of topological and measure-theoretic analogies have a game of Mazur as their common origin. (cf. [4], [5], [6])

Definition. A basic system is called a complete system if it satifies the condition

(δ) Every sequence $\langle A_n \rangle_{n \in \mathbb{N}}$ of regions, for which the sequence $\langle \psi_n(A_n) \rangle_{n \in \mathbb{N}}$ is descending, has a nonempty intersection.

Example C. Let (X, d) be a metric space, let $\mathcal{C}$ be the family of all closed spheres in X, and, for each region $A \in \mathcal{C}$ and each $n \in \mathbb{N}$, let $\psi_n(A)$ be the first subregion B of A with $\operatorname{diam}(B) \leq \frac{1}{n}$. Then the characterization (m) is equivalent to the condition (δ).

Example D. Let $(Y, <)$ be an ordered set with no smallest or largest elements and containing a denumerable everywhere dense set, let X be a subset of Y, let $\mathcal{C}$ be the family of all bounded, closed intervals in X, and let ψ_n be the identity mapping for each $n \in \mathbb{N}$. Then the characterization (o) is equivalent to the condition (δ).

Example E. Let $(X, \mathcal{T})$ be a topology, let $\mathcal{C}$ be the family of all closed sets, and let ψ_n be the identity mapping for each $n \in \mathbf{N}$. Then the condition (δ) is equivalent to the topology being countably compact.

The subsets of a set X are classified relative to a given family $\mathcal{C}$ in the following manner: A set is a singular set if every region has a subregion disjoint from the set. A set is a meager set if it is representable as a countable union of singular sets. A set which is not meager is called an abundant set. In the case that $(X, \mathcal{C})$ is a topology, the singular, meager, and abundant sets coincide with the nowhere dense, first category, and second category sets. Generalizing the notion of a Baire topology, we say $(X, \mathcal{C})$ is a Baire family if every region is an abundant set.

The importance of complete systems stems from the following set-theoretical formulation of the Baire Category Theorem (cf. [6] p. 71).

Theorem 1. If $\mathfrak{X}$ is a complete system then $(X, \mathcal{C})$ is a Baire family.

This theorem encompasses numerous topological versions of the classical Baire Category Theorem, including the following examples of complete systems.

Example F. (X, d) is a complete metric space and $\mathcal{C}$ is the family of all open sets in X. For each region A and each $n \in \mathbf{N}$ we take $\psi_n(A)$ to be the first region B in $\mathcal{C}$ whose closure is contained in A and with $\text{diam}(B) \leq \frac{1}{n}$.

Example G. $(X, \mathcal{C})$ is a countably compact, regular topology (cf. [1] 29.25). For each region A and each $n \in \mathbf{N}$, we use the assumed regularity to define $\psi_n(A)$ to be the first region in $\mathcal{C}$ whose closure is contained in A. The satisfaction of the condition (δ) results from applying the assumed countable compactness to the sequence of closures of the sets $\psi_n(A_n)$.

Example H. $(X, \mathcal{C})$ is a locally compact, regular topology (cf. [1] 31.27). For each region A and each $n \in \mathbf{N}$ we define $\psi_n(A)$ to be the first region in $\mathcal{C}$ whose closure is a compact subset of A.

Example I. $(X, \mathcal{C})$ is Smirnov's deleted sequence topology; i.e. $X = \mathbf{R}$ and $\mathcal{C}$ consists of all sets A representable in the form $A = G - E$, where G is an open set in the usual topology for $\mathbf{R}$ and $E \subset \{\frac{1}{m} : m \in \mathbf{N}\}$. For each such region A and each $n \in \mathbf{N}$ we define $\psi_n(A)$ to be the first open interval B whose closure is contained in A with length $1(B) \leq \frac{1}{n}$. We note that this topology is not regular, not countably compact, not locally compact, etc. (cf. [8] No. 64).

Example J. $(X, \mathcal{C})$ is the countable complement topology for an uncountable set X and all the mappings ψ_n are the identity mapping (cf. [8] No. 20).

We now show how a given basic system $\mathfrak{X} = \langle X, \mathcal{C}, \Psi \rangle$ generates a second basic system $\mathfrak{X}^* = \langle X^*, \mathcal{C}^*, \Psi^* \rangle$ by means of a purely set-theoretical construction.

Definition. A sequence $\langle E_m \rangle_{m \in \mathbf{N}}$ of regions is called a regular sequence if it satisfies the condition

$$E_{m+1} \subset \psi_m(E_m)$$

for every $m \in \mathbf{N}$.

We note that every regular sequence is necessarily a descending sequence. We also note that every region A contains a regular sequence; e.g. take $E_1 = A$, $E_2 = \psi_2\psi_1(A)$, $E_3 = \psi_3\psi_2\psi_2\psi_1(A)$, and, in general , $E_m = \psi_m\psi_{m-1}\psi_{m-1}\cdots\psi_2\psi_2\psi_1(A)$ for $m > 2$.

Definition. Sequences of sets $\langle E_m \rangle_{m \in \mathbf{N}}$ and $\langle F_m \rangle_{m \in \mathbf{N}}$ are said to be interlaced if $(\forall n)(\exists m)(E_m \subset F_n)$ and $(\forall m)(\exists k)(F_k \subset E_m)$.

Starting with a given basic system $\mathfrak{X} = \langle X, \mathcal{C}, \Psi \rangle$ we identify regular sequences of regions which are interlaced and thereby obtain an equivalence relation on the set of all regular sequences. The equivalence class containing a given regular sequence $\langle E_m \rangle_{m \in \mathbf{N}}$ will be denoted by $[\langle E_m \rangle]$. The set of all such equivalence classes is denoted by X^*. For each $A \in \mathcal{C}$ we define

$$A^* = \{[\langle E_m \rangle] \in X^* : (\exists m \in \mathbf{N})(E_m \subset A)\};$$

that is A^* consists of all equivalence classes $x^* \in X^*$ having the property that for any representative element $\langle E_m \rangle_{m \in \mathbf{N}}$ in x^*, there exists an index m such that $E_m \subset A$. For each $A \in \mathcal{C}$ and each $n \in \mathbf{N}$ we define

$$\psi_n^*(A^*) = (\psi_n(A))^*$$

Placing $\mathcal{C}^* = \{A^* : A \in \mathcal{C}\}$ and $\Psi^* = \langle \psi_n^* : n \in \mathbf{N} \rangle$ we then have a well-defined basic system $\mathfrak{X}^* = \langle X^*, \mathcal{C}^*, \Psi^* \rangle$.

Definition. A regular sequence $\langle E_m \rangle_{m \in \mathbf{N}}$ of regions in $\mathcal{C}$ is said to converge to a point $x \in X$ if its intersection is $\{x\}$; in which case x is called the limit of the given sequence.

Definition. A basic system $\mathfrak{X}$ is called a point-regular system if the following conditions are satisfied:

(1) Every point is the limit of a regular sequence of regions.

(2) If $\langle E_m \rangle_{m \in \mathbf{N}}$ is any regular sequence of regions converging to a point x and A is any region containing x then there exists an index m such that $E_m \subset A$.

Note that condition (2) implies any two regular sequences converging to the same point are interlaced.

Example A is point-regular with each point $x \in X$ being the limit of the regular sequence $\langle E_{m,x} \rangle_{m \in \mathbf{N}}$ of open spheres

$$E_{m,x} = \{y \in X : d(x,y) < \frac{1}{2m}\}.$$

Example B is point-regular with each point $x \in X$ being the limit of the regular sequence $\langle E_{m,x} \rangle_{m \in \mathbf{N}}$ of regions

$$E_{m,x} = (a_{m,x}, b_{m,x}) \cap X$$

defined inductively as follows: $a_{1,x}$ is the first element r_{j_1} in the enumeration of Q with index $j_1 > 2$ satisfying $r_{j_1} < x$ and $b_{1,x}$ is the first element r_{k_1} of Q with index $k_1 > 2$ satisfying $x < r_{k_1}$; for $m > 1$, $a_{m,x}$ is the first element r_{j_m} with index $j_m > j_{m-1}$ satisfying $a_{m-1,x} < r_{j_m} < x$ and $b_{m,x}$ is the first element r_{k_m} with index $k_m > k_{m-1}$ satisfying $x < r_{k_m} < b_{m-1,x}$. We then have $\psi_m(E_{m,x}) = E_{m,x}$ for all $m \in \mathbf{N}$ and all $x \in X$.

The proof of the following fact is straight-forward.

Lemma. If $\mathfrak{X}$ is point-regular, A^* and B^* are regions in $\mathcal{C}^*$, and $A^* \subset B^*$ then $A \subset B$.

Definition. A basic system $\mathfrak{X}$ is called idempotent if each mapping ψ_n is idempotent; i.e. if $\psi_n \psi_n = \psi_n$ for every $n \in \mathbf{N}$.

The systems of both Example A and Example B are idempotent.

Theorem 2. If $\mathfrak{X}$ is a point-regular, idempotent system then $\mathfrak{X}^*$ is a complete system.

Proof. Suppose $\langle A_n^* \rangle_{n \in \mathbf{N}}$ is a sequence of regions in $\mathcal{C}^*$ for which the sequence $\langle \psi_n^*(A_n^*) \rangle_{n \in \mathbf{N}}$ is descending. By virtue of the foregoing lemma, the sequence $\langle \psi_m(A_m) \rangle_{m \in \mathbf{N}}$ is a descending sequence of regions in $\mathcal{C}$. According to the idempotence assumption, this sequence is a regular sequence. It follows that $[\langle \psi_m(A_m) \rangle] \in \psi_n^*(A_n^*)$ for every $n \in \mathbf{N}$. This implies the intersection of the sets A_n^* is nonempty.

Definition. A complete system $(Y, \mathcal{D}, \Phi)$ is called a completion of a basic system $(X, \mathcal{C}, \Psi)$ if there exists a one-to-one function $\iota : X \to Y$ having the property that for every region $B \in \mathcal{D}$ there exists a region $A \in \mathcal{C}$ such that $\iota(A) \subset B$.

Theorem 3. If $\mathfrak{X}$ is a point regular, idempotent system then $\mathfrak{X}^*$ is a completion of $\mathfrak{X}$.

Proof. For each point $x \in X$ we choose a regular sequence $\langle E_{m,x}\rangle_{m\in\mathbf{N}}$ converging to x and define $\iota(x) = [\langle E_{m,x}\rangle]$. The function ι is a well-defined one-to-one mapping of X into X^*. Using condition (2) in the definition of a point-regular system it is readily seen that $\iota(A) \subset A^*$ for every region $A \in \mathcal{C}$.

Application 1. Hausdorff's completion of a metric space.

We start with a given metric space (X, d) and specialize Theorem 3 to Example A. For elements $x^* = [\langle E_m\rangle]$, $y^* = [\langle F_m\rangle]$ in X^* we define

$$d^*(x^*, y^*) = \lim_{m\to\infty} d(E_m, F_m).$$

(a) (X^*, d^*) is a pseudometric space.

Proof. The sequence $\langle d(E_m, F_m)\rangle_{m\in\mathbf{N}}$ is a monotone increasing sequence of non-negative real numbers bounded above by $d(x_2, y_2) + 2$, where $x_2 \in E_2$ and $y_2 \in F_2$. The limit thus exists and, because of the interlacing, is independent of the representative sequences $\langle E_m\rangle_{m\in\mathbf{N}}$ and $\langle F_m\rangle_{m\in\mathbf{N}}$ selected. We obviously have

$$d^*(x^*, x^*) = 0 \quad\text{and}\quad d^*(x^*, y^*) = d^*(y^*, x^*)$$

while the triangle inequality

$$d^*(x^*, y^*) \le d^*(x^*, z^*) + d^*(z^*, y^*)$$

is a consequence of the inequality

$$d(E_m, F_m) \le d(E_m, G_m) + d(G_m, F_m) + \operatorname{diam}(G_m).$$

(b) The mapping $\iota : X \to X^*$ is an isometry.

Proof. Suppose $\langle E_m\rangle_{m\in\mathbf{N}}$, $\langle F_m\rangle_{m\in\mathbf{N}}$ are regular sequences of regions in $\mathcal{C}$ converging to points $x, y \in X$, respectively. For each index $m > 1$ and any points $u \in F_m$, $v \in F_m$ we have

$$d(x, y) \le d(x, u) + d(u, v) + d(v, y) \le d(u, v) + \frac{2}{m-1}$$

whence

$$d(x,y) - \frac{2}{m-1} \leq d(u,v)$$

This implies

$$d(x,y) - \frac{2}{m-1} \leq d(E_m, F_m) \leq d(x,y)$$

since $x \in E_m$ and $y \in F_m$. Taking limits yields

$$d^*(\iota(x), \iota(y)) = d(x,y)$$

(c) If S is an open sphere in X^* then there is a region $A \in \mathcal{C}$ such that $A^* = S$.

Proof. Let $S = \{y^* \in X^* : d^*(x^*, y^*) < r\}$ be an open sphere in X^* with center $x^* = [\langle E_m \rangle]$ in X^*. Set

$$A = \bigcup \{C \in \mathcal{C} : \iota(C) \subset S\}.$$

Since A is an open set in X, we have only to show A is non-empty to establish that A is a region in $\mathcal{C}$.

Choose $m \in \mathbf{N}$ so that $\operatorname{diam}(E_m) < \frac{r}{2}$. Suppose u is any point of E_m. Let $\langle G_n \rangle_{n \in \mathbf{N}}$ be a regular sequence of regions in $\mathcal{C}$ converging to u. Then there exists $n_0 \in \mathbf{N}$ such that $n_0 \geq m$ and $G_n \subset E_m$ for all $n \geq n_0$. For such n, if $s \in E_n$ and $t \in G_n$ then $s, t \in E_m$ and accordingly $d(s,t) < \frac{r}{2}$. This implies

$$d^*(x^*, \iota(u)) = d^*([\langle E_n \rangle], [\langle G_n \rangle]) = \sup_{n \geq n_0} d(E_n, G_n) < r.$$

Hence, $\iota(u)$ is an element of S. We thus see that, for $C = E_m$, we have $\iota(C) \subset S$. We conclude A is a region in $\mathcal{C}$.

Assume $z^* = [\langle F_m \rangle] \in A^*$. Then there exists $m \in \mathbf{N}$ such that $F_n \subset A$, and consequently $\iota(F_n) \subset S$, for all indices $n \geq m$. For such n, choose a point $w \in F_n$. Let $\langle G_j \rangle_{j \in \mathbf{N}}$ be a regular sequence of regions in $\mathcal{C}$ converging to w. Then there is an index $j_n \geq n$ such that $G_j \subset F_n$ for all $j \geq j_n$. Because $\iota(w)$ is an element of S,

$$\lim_{j \to \infty} d(E_j, G_j) = d^*(x^*, \iota(w)) < r.$$

For each $n \geq m$ and every $j \geq j_n$ we have $d(E_n, F_n) \leq d(E_j, G_j)$, whence $\lim_{n \to \infty} d(E_n, F_n) < r$. Therefore, $z^* \in S$.

Conversely, assume $z^* = [\langle F_m \rangle] \in S$. Let $q = d^*(x^*, z^*)$. Choose $m_0 \in \mathbf{N}$ so that $\operatorname{diam}(F_{m_0}) < \frac{r-q}{3}$. Suppose u is any point of F_{m_0} and let $\langle G_j \rangle_{j \in \mathbf{N}}$ be a regular sequence of regions in $\mathcal{C}$ converging to u. Then there is an index $j_0 \geq m_0$ such that $G_j \subset F_{m_0}$ for all $j \geq j_0$. For these indices j,

$$\begin{aligned} d(E_j, G_j) &\leq d(E_j, F_j) + d(F_j, G_j) + \operatorname{diam}(F_j) \\ &\leq q + \frac{r-q}{3} + \frac{r-q}{3} < r. \end{aligned}$$

Hence,

$$d^*(x^*, \iota(u)) = \lim_{j\to\infty} d(E_j, G_j) < r.$$

For every point $u \in F_{m_0}$ we thus have $\iota(u) \in S$. This means $F_{m_0} \subset A$. Therefore, $z^* \in A^*$.

We conclude $A^* = S$.

(d) (X^*, d^*) is complete.

Proof. Let $\langle B_p\rangle_{p\in\mathbb{N}}$ be a descending sequence of closed spheres in X^* whose diameters converge to 0; say

$$B_p = \{y^* \in X^* : d^*(x_p^*, y^*) \leq r_p\}$$

where $x_p^* \in X^*$ and $r_p > 0$. In view of the characterization (m) we have only to establish $\bigcap_{p=1}^{\infty} B_p \neq \emptyset$.

Choose an increasing sequence $\langle p_n\rangle_{n\in\mathbb{N}}$ of natural numbers such that $\operatorname{diam}(B_{p_n}) < \frac{1}{n}$ for each $n \in \mathbb{N}$ and set

$$S_n = \{y^* \in X^* : d^*(x_{p_n}^*, y^*) < r_{p_n}\}.$$

According to property (c), for each $n \in \mathbb{N}$ there is a region A_n in $\mathcal{C}$ with $A_n^* = S_n$. If $u, v \in A_n$ then $\iota(u), \iota(v) \in S_n$ and, ι being an isometry, $d(u,v) = d^*(\iota(u), \iota(v)) < \frac{1}{n}$. Therefore, $\psi_n(A_n) = A_n$ for every n.

The sequence $\langle S_n\rangle_{n\in\mathbb{N}}$ is a descending sequence of regions in $\mathcal{C}^*$ with $\psi_n^*(S_n) = S_n$ for each $n \in \mathbb{N}$. By virtue of the completeness of $\mathfrak{X}^*$, there exists an element of X^* which belongs to every set S_n and hence belongs to every set B_p.

(e) The metric space induced by the pseudo-metric space (X^*, d^*) is Hausdorff's completion of (X, d).

Proof. A completion of a pseudometric (resp., metric) space (X, d) is a complete pseudometric (resp., metric) space having a dense subset isometric to X and such completions are uniquely determined up to isometries. By virtue of property (c), (X^*, d^*) is a completion of (X, d). The metric space induced by the complete pseudometric space (X^*, d^*) (by identifying elements $x^*, y^* \in X^*$ with $d^*(x^*, y^*) = 0$) is a complete metric space having a dense subset isometric to X and is isometric to Hausdorff's completion of (X, d).

Application 2. Dedekind's completion of the rational numbers.

We start with an ordered set $(X, <)$ in the context of Example B with $Q \subset X$. For elements $x^* = [\langle E_m \rangle]$, $y^* = [\langle F_n \rangle]$ in X^* we define

$$x^* <^* y^* \iff (\exists m)(\exists n)(\forall u \in E_m)(\forall v \in F_n)(u < v).$$

(a) $(X^*, <^*)$ is an ordered set.

Proof. It is a simple matter to verify the relation $<^*$ is well-defined, irreflexive, and transitive. It remains to show that, for any elements $x^*, y^* \in X^*$, either $x^* <^* y^*$, $y^* <^* x^*$, or $x^* = y^*$. We assume neither $x^* <^* y^*$ nor $y^* <^* x^*$ holds. Then for all natural numbers m, n we have $E_m \cap F_n \neq \emptyset$.

Suppose $n \in \mathbb{N}$ is given and $F_n = (r, s) \cap X$, with $r, s \in Q$. Let $r = r_{n_1}$ and $s = r_{n_2}$ in the enumeration of Q. Place $n_0 = \max\{n, n_1, n_2\}$ and let $\psi_{n_0}(F_{n_0}) = J_{n_0} \cap X$, where $J_{n_0} = (r_{n_3}, r_{n_4})$ for r_{n_3}, r_{n_4} enumerated elements of Q. From the inclusions $F_{n_0+1} \subset \psi_{n_0}(F_{n_0}) \subset F_{n_0} \subset F_n$ and the fact that none of the elements $r_1, r_2, \ldots, r_{n_0+1}$ is an endpoint of J_{n_0} we obtain $r < r_{n_3} < r_{n_4} < s$ and $F_{n_0+1} \subset (r_{n_3}, r_{n_4})$. Place $m_0 = \max\{n_0, n_3, n_4\}$ and let $\psi_{m_0}(E_{m_0}) = I_{m_0} \cap X$, where $I_{m_0} = (p, q)$ with $p, q \in Q$. Suppose we assume $r \in I_{m_0}$. Then, since I_{m_0} contains at most one of the elements $r_1, r_2, \ldots, r_{m_0+1}$, we must have $r_{n_3} \notin I_{m_0}$. The inclusions $E_{m_0+1} \subset \psi_{m_0}(E_{m_0}) \subset (p, r_{n_3})$ then yield the contradiction $E_{m_0+1} \cap F_{m_0+1} = \emptyset$. We thus see $r \notin I_{m_0}$. Similarly, $s \notin I_{m_0}$. Now, $(p, q) \cap (r, s) \neq \emptyset$, $r \notin (p, q)$, and $s \notin (p, q)$ imply $(p, q) \subset (r, s)$ and consequently $E_{m_0} \subset F_n$. Hence, for every $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $E_m \subset F_n$. Similarly, for every $m \in \mathbb{N}$ there exists $k \in \mathbb{N}$ such that $E_m \subset F_n$. Similarly, for every $m \in \mathbb{N}$ there exists $k \in \mathbb{N}$ such that $F_k \subset E_m$. We conclude $x^* = y^*$.

(b) The mapping $\iota : X \to X^*$ is an isomorphism.

Proof. Suppose $x, y \in X$ and $x < y$. Let $\iota(x) = [\langle E_m \rangle]$ and $\iota(y) = [\langle F_n \rangle]$, where $\langle E_m \rangle_{m \in \mathbb{N}}$ and $\langle F_n \rangle_{n \in \mathbb{N}}$ are regular sequences of regions in $\mathcal{C}$ converging respectively to x and y. Choose $r, s, t \in Q$ so that $r < x < s < y < t$. Then $I = (r, s) \cap X$ and $J = (s, t) \cap X$ are regions containing x and y. Accordingly, there exists $m, n \in \mathbb{N}$ such that $E_m \subset I$ and $F_n \subset J$. This implies $\iota(x) <^* \iota(y)$.

(c) The set $\iota(Q)$ is everywhere dense in X^*.

Proof. Suppose $x^* = [\langle E_m \rangle]$, $y^* = [\langle F_n \rangle]$ are elements of X^* with $x^* <^* y^*$. Choose $m_0, n_0 \in \mathbb{N}$ such that for all $x \in E_{m_0}$ and all $y \in F_{n_0}$ we have

$x < y$. The element $a = \sup E_{m_0}$ belongs to Q. Hence, there is an element r_{m_1} in the enumeration of Q such that $a = r_{m_1}$. Let $m_2 = \max\{m_0, m_1\}$ and let $c = \sup \psi_{m_2}(E_{m_2})$. We have $c \in Q$ and $c < a$. Choose $r \in Q$ satisfying $c < r < a$ and let $\langle G_k \rangle_{k \in \mathbf{N}}$ be a regular sequence of regions in $\mathcal{C}$ converging to r. Then there is an index k_0 such that $G_{k_0} \subset (c, a)$. For all $u \in E_{m_2+1}$, all $z \in G_{k_0}$, and all $v \in F_{m_0}$ we have $u < z < v$. This means $x^* <^* \iota(r) <^* y^*$.

(d) If S is a nonempty open interval in X^* with endpoints in $\iota(Q)$ then $A = \iota^{-1}(S)$ is a region in $\mathcal{C}$ with $A^* = S$.

Proof. Let $S = (r^*, s^*)$, where $r^* = [\langle C_j \rangle]$, $s^* = [\langle D_k \rangle]$ are elements of $\iota(Q)$ with $r^* <^* s^*$. Let $r, s \in Q$ be such that $r^* = \iota(r)$ and $s^* = \iota(s)$. It being clear that A is a region in $\mathcal{C}$, we have only to establish $A^* = S$.

Suppose $x^* = [\langle E_m \rangle]$ is an element of A^*. Then there exists $m \in \mathbf{N}$ for which $E_m \subset A$. Due to the regularity of the sequence $\langle E_m \rangle_{m \in \mathbf{N}}$ and the particular manner of defining the mappings ψ_n, we can find points $p, q \in Q$ and an index $n > m$ such that $r < p < q < s$ and $E_n \subset (p, q)$. Let $a, b \in Q$ satisfy $a < r < p < q < s < b$. Since there are regular sequences of regions in the equivalence classes r^* and s^* converging to r and s, respectively, there are indices $j, k \in \mathbf{N}$ such that $C_j \subset (a, p)$ and $D_k \subset (q, b)$. From $E_n \subset (p, q)$ we then perceive $r^* <^* x^* <^* s^*$, so $x^* \in S$.

Conversely, suppose $x^* \in S$. Then there exist $j, k, m \in \mathbf{N}$ such that

$$(\forall u \in C_j)(\forall w \in E_m)(u < w) \quad \text{and} \quad (\forall w \in E_m)(\forall v \in D_k)(w < v).$$

In conjunction with the facts that $r \in C_j$ and $s \in D_k$, this implies $E_m \subset A$. We conclude $x^* \in A^*$.

(e) $(X^*, <^*)$ is the completion of $(X, <)$.

Proof. In view of Cantor's theorem that any two complete ordered sets having neither smallest nor largest elements and containing everywhere dense, denumerable subsets are isomorphic, we have only to verify $(X^*, <^*)$ is complete.

Suppose $\langle B_m \rangle_{m \in \mathbf{N}}$ is a descending sequence of closed intervals $B_m = [a_m^*, b_m^*]$ with $a_m^*, b_m^* \in X^*$. We show $\bigcap_{m=1}^{\infty} B_m \neq \emptyset$. Without loss of generality, we assume all elements a_m^* are different, all elements b_m^* are different, and none of the elements $r_k^* = \iota(r_k)$ belongs to every set B_m, where $\langle r_k \rangle_{k \in \mathbf{N}}$ is a fixed enumeration of the set Q.

Choose $m_1 \in \mathbf{N}$ so that $r_1^*, r_2^* \notin B_{m_1}$. Continuing inductively, we choose $m_k \in \mathbf{N}$ so that $m_k > m_{k-1}$ and $r_{k+1}^* \notin B_{m_k}$, for each $k > 1$. Choose sequences $\langle p_k^* \rangle_{k \in \mathbf{N}}$, $\langle q_k^* \rangle_{k \in \mathbf{N}}$ of elements of $\iota(Q)$ satisfying the relationships

$$a_{m_k}^* <^* p_k^* <^* a_{m_{k+1}}^* <^* b_{m_{k+1}}^* <^* q_k^* <^* b_{m_k}^*.$$

We note that none of the elements $r_1^*, r_2^*, \ldots, r_{k+1}^*$ belongs to the open interval $S_k = (p_k^*, q_k^*)$, nor is any one of these elements an endpoint of S_k. Then $\langle S_k \rangle_{k \in \mathbb{N}}$ is a descending sequence of regions in $\mathcal{C}^*$ with

$$\psi_k^*(S_k) = (\psi_k(\iota^{-1}(S_k)))^* = (\iota^{-1}(S_k))^* = S_k.$$

By completeness of $\mathfrak{X}^*$, there is an element of X^* which belongs to every set S_k and hence belongs to every set B_m.

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