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Special Subsets of the Real Line

We introduce two properties $\mathcal{M}$ and $\mathcal{F}$ of subsets of the real line $\mathbf{R}$ concerning the control measure problem. We present a topological point of view on the problem. We also make remarks on above properties under the assumption $MA + \neg CH$.

We use the topological definitions in [E], the Boolean measure definitions as in [F], and the set theoretic ones in [J]. We use standard notation, in particular MA, $\neg CH$ abbreviate statements: Martin's Axiom, and negation of the Continuum Hypothesis. The symbols $\bigwedge, \bigvee, \triangle$ denote infimum, supremum and symmetric difference in Boolean algebras. By $\mathcal{P}(X), \mathcal{B}(X)$ we mean the power set of X and the set of all Borel subsets of $X \subseteq \mathbf{R}$.

Let $\mathcal{A}$ be a σ-algebra of subsets of a set X. A Maharam submeasure ν on $\mathcal{A}$ is a map from $\mathcal{A}$ to $[0,1]$ which satisfies the following properties:

(1) $\nu(\emptyset) = 0$.

(2) If $E, F \in \mathcal{A}$ and $E \subseteq F$, then $\nu(E) \leq \nu(F)$.

(3) If $E, F \in \mathcal{A}$, then $\nu(E \cup F) \leq \nu(E) + \nu(F)$.

(4) If $\{E_n\}_{n\in\omega}^{\infty}$ is a sequence of sets from $\mathcal{A}$ and $\bigcap_{k\in\omega} \bigcup_{n\geq k} (E_n \triangle E) = \emptyset$, then $\lim_{n\to\infty} \nu(E_n) = \nu(E)$.

(5) $\nu(\{x\}) = 0$ for all $x \in X$.

An important problem of measure theory is the following control measure problem: for given a σ-algebra $\mathcal{A}$ and a Maharam submeasure ν on $\mathcal{A}$, does there exist a probability measure μ on $\mathcal{A}$, such that for all $A \in \mathcal{A}$, $\nu(A) = 0$ if and only if $\mu(A) = 0$?

By $\mathcal{C}$ and $\mathcal{N}$ we mean the σ-ideal of strong measure zero subsets of the real line $\mathbf{R}$ and the σ-ideal of universally measure zero subsets of $\mathbf{R}$, respectively; see [Mi] for definitions.

We introduce the following σ-ideal $\mathcal{M}$: A set $X \subseteq \mathbf{R}$ belongs to $\mathcal{M}$ if and only if for every Maharam submeasure ν on the σ-field $\mathcal{B}(X)$ of Borel subsets of X (or equivalently on $\mathcal{B}(\mathbf{R})) \nu(X) = 0$.

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For every Maharam submeasure ν on $\mathcal{B}(\mathbf{R})$ the sets of ν submeasure zero form a σ-ideal I_ν (because (3) and (4) imply $\nu(\bigcup_{n\in\omega}E_n) \leq \sum_{n\in\omega}\nu(E_n)$). Hence $\mathcal{M} = \bigcap\{I_\nu : \nu$ is a Maharam submeasure on $\mathcal{B}(\mathbf{R})\}$ is a σ-ideal.

Lemma 1

$$\mathcal{C} \subseteq \mathcal{M} \subseteq \mathcal{N}$$

PROOF. The second inclusion is obvious. We prove the first inclusion. For each $x \in X$ and any Maharam submeasure ν we have $\nu(\{x\}) = 0$ and $\{x\} = \bigcap_{n\in\omega}U_n$, where $\{U_n\}_{n\in\omega}$ is monotonically decreasing sequence of intervals. Hence for every $\epsilon > 0$ there exists $\delta > 0$ such that: if U is any subinterval of $[0,1]$ of diameter less then δ, then $\nu(U) < \epsilon$.

Therefore it is easy to see that any strong measure zero set is a Maharam submeasure ν zero set. □

Remark 1 *An example of a set from $\mathcal{N}\setminus\mathcal{M}$ answers the control measure problem.*

Corollary 1 *If $MA + \neg CH$ is true then:*

(i) *For every set $X \subseteq \mathbf{R}$ of cardinality $< 2^\omega$ $X \in \mathcal{M}$.*

(ii) *There exists a set $X \subseteq \mathbf{R}$ of cardinality 2^ω such that $X \in \mathcal{M}$.*

PROOF. For (i) see [L] and lemma 1. For (ii): the example of such X is the generalized Lusin set; see [L]. □

By a good σ-ideal I we always mean a σ-ideal on $\mathcal{B}(X)$, where $X \subseteq \mathbf{R}$, such that I contains all points of X and the quotient algebra $\mathcal{B}(X)/I$ is atomless and satisfies the countable chain condition.

We introduce a topology $\bar{\tau}$ in the algebra $\mathcal{B}(X)/I$ by (the condition): the sequence $\{x_n\}_{n\in\omega}$ of elements of $\mathcal{B}(X)/I$ converges to an element x of $\mathcal{B}(X)/I$ if and only if $\bigwedge_{k\in\omega}\bigvee_{n\geq k}(x_n\triangle x) = I$ (in symbols $x_n \Rightarrow x$). We call a set U open if whenever $x \in U$ and $x_n \Rightarrow x$, then $x_n \in U$ for n sufficiently large. For the details and for the connections with the topology τ on $\mathcal{B}(X)$ introduced in the same way by the convergence $x_n \to x$ if and only if $\bigcap_{k\in\omega}\bigcup_{n\geq k}(x_n\triangle x) = \emptyset$ see [G].

Remark 2 *Since $MA + \neg CH$ implies $\mathcal{B}(X) = \mathcal{P}(X)$ for all $X \subseteq \mathbf{R}$ of uncountable cardinality $< 2^\omega$, $Con(ZFC + MA + \neg CH_+$ there exists a good σ-ideal I on $\mathcal{B}(X)$*

for a set X of uncountable cardinality $< 2^\omega$) is equivalent to $Con(ZFC_+$ there exists a measurable cardinal).

Lemma 2 *For any good σ-ideal I on $\mathcal{B}(X)$ we have:*

(i) *The space $\mathcal{B}(X)/I$ with the topology $\bar{\tau}$ is a sequential space.*

(ii) *If $MA + \neg CH$ is true and a set X is of cardinality $< 2^\omega$, then $\mathcal{B}(X)/I$ is a Frechet space.*

PROOF. See [G] lemmas 1 and 3. □

Remark 3 *The lemma 2(ii) implies that $\mathcal{B}(X)/I$ is a complete, weakly countably distributive, ccc Boolean algebra. See [G].*

Lemma 3 *For any good σ-ideal I on $\mathcal{B}(X)$ the following are equivalent:*

(i) *There exists a strictly positive Maharam submeasure on the complete algebra $\mathcal{B}(X)/I$.*

(ii) *The topology $\bar{\tau}$ on the algebra $\mathcal{B}(X)/I$ is metrizable.*

(iii) *The space $\mathcal{B}(X)/I$ with the topology $\bar{\tau}$ and with the operation $\triangle$ is a topological group.*

PROOF. For *(i)* $\equiv$ *(ii)* see [M]. For *(ii)* $\equiv$ *(iii)* see [S-I] and [V] Thm. 14 p. 135. □

By the above results, we have the following topological characterization of sets from $\mathcal{M}$:

Lemma 4 *A set $X \in \mathcal{M}$ if and only if for every good σ-ideal I the topology $\bar{\tau}$ in $\mathcal{B}(X)/I$ is nonmetrizable if and only if the algebra $\mathcal{B}(X)/I$ is not a topological group in the topology $\bar{\tau}$ and with the operation $\triangle$.* □

Lemma 5 *For any good σ-ideal I the algebra $\mathcal{B}(X)/I$ is a regular space (with the topology $\bar{\tau}$) if and only if it is a Hausdorff space.*

PROOF. See de Marr in [B] p. 245 and [V] Thm. 14 p. 135. □

Remark 4 *For algebras with the order sequential topology (i.e. os-topology; see [V] for definition), which is not ccc the above is not true; see [G] Prop. 2 (iii).*

Lemma 6 *If $x_n \Rightarrow x$ and $y_n \Rightarrow y$, then $x_n \triangle y_n \Rightarrow x \triangle y$.* $\square$

Now we can prove:

Proposition 1 *If $MA + \neg CH$ is true, then the topology $\bar{\tau}$ in $\mathcal{B}(X)/I$ is not Hausdorff for any good σ-ideal $I \subseteq \mathcal{B}(X)$, where $X \subseteq \mathbf{R}$ is of cardinality $< 2^\omega$.*

PROOF. Assume that $\mathcal{B}/(X)I$ is a Hausdorff space. By lemma 5 it is a regular space, and by Proposition 2(iv) and Lemma 3 from [G], it is a locally sequentially compact (see [E] for the definition), sequential space. This implies the product $\mathcal{B}(X)/I \times \mathcal{B}(X)/I$, with the product topology, is a sequential space (see Boehme in [E] p. 271). It is not difficult to show that the operation $\triangle$ from $\mathcal{B}(X)/I \times \mathcal{B}(X)/I$ is a continuous function — use the Lemma 2 from [G], lemma 6. Hence $\mathcal{B}(X)/I$ is topological group, but this is not possible because $X \in \mathcal{M}$ (see lemma 4). $\square$

We say $X \subseteq \mathbf{R}$ has the property $\mathcal{F}$ if for every good σ-ideal I the algebra $\mathcal{B}(X)/I$ is not a Hausdorff space with the topology $\bar{\tau}$. It is obvious that $\mathcal{F} \subseteq \mathcal{M}$.

By proposition 1 we have:

Corollary 2 *If $MA + \neg CH$ is true, then every set $X \subseteq \mathbf{R}$ of cardinality $< 2^\omega$ has the property $\mathcal{F}$.* $\square$

The method used in the proof of Proposition 1 gives:

Proposition 2 *Let I be a good σ-ideal on $\mathcal{B}(X)$. There exists a strictly positive Maharam submeasure on $\mathcal{B}(X)/I$ if and only if the product $\mathcal{B}(X)/I \times \mathcal{B}(X)/I$, with the product topology of topology $\bar{\tau}$ is a sequential space.* $\square$

Corollary 3 *(i) If $\mathcal{B}(X)/I$ is a Hausdorff, sequentially compact space, then there exists a strictly positive Maharam submeasure on $\mathcal{B}(X)/I$.*

(ii) There exists a strictly positive Maharam submeasure on $\mathcal{B}(X)/I$ if and only if $\mathcal{B}(X)/I$ has a countable basis of the neighborhoods of zero. $\square$

Compare with Theorem 3 in [M]. Corollary 3(ii) is true for every complete, atomless, ccc, Boolean algebra.

The existence of a Hausdorff, sequentially compact $\mathcal{B}(X)/I$ space gives a negative answer to the control measure problem.

PROBLEM: Let I be a good ideal in $\mathcal{B}(X)$. If $\mathcal{B}(X)/I$ is a Hausdorff space with the topology $\bar{\tau}$, then is there a strictly positive Maharam submeasure on $\mathcal{B}(X)/I$?

Corollary 4 *If $I \subseteq I^*$ are good ideals on $\mathcal{B}(X)$ and there exists a strictly positive Maharam submeasure on $\mathcal{B}(X)/I$, then there exists a strictly positive Maharam submeasure on $\mathcal{B}(X)/I^*$.*

PROOF. The natural mapping from $\mathcal{B}(X)/I$ onto $\mathcal{B}(X)/I^*$ is open. □

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