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The Relations of Hausdorff, $*$–Hausdorff, and Packing Measures

1. Introduction

We use several distinct outer measures to investigate the size of thin sets in $\mathbf{R}^d$. In this paper, we are interested in only three outer measures, ϕ–m (Hausdorff measure), ϕ–m^* ($*$–Hausdorff measure), and ϕ–p (packing measure), based on a monotone function ϕ.

In [5], it was shown that $\phi - m(E) \leq \phi - m^*(E) \leq \phi - p(E)$ for any set $E \subset \mathbf{R}^d$.

We will investigate some other relations of the aforementioned outer measures.

We adopt a new definition of $*$–regularity, and p–regularity by using $*$–Hausdorff and packing measures, as regularity is defined by using Hausdorff measure. We will show that the decomposition theorem holds for packing measure as it does for Hausdorff measure [1]. Further, every subset of a p–regular set has identical $*$–Hausdorff and packing measures, whereas every subset having positive packing measure of a p–irregular set cannot have identical $*$–Hausdorff and packing measures. Futhermore, the set of p–regular points of a given set is contained in the set of $*$–regular points of the given set ϕ–p.a.s., and their difference is of ϕ–m measure zero. A similar result is

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obtained for the set of p–regular points and the set of regular points of the given set. If E is ϕ–p–measurable and ϕ–$p(E) < \infty$ for $\phi(t) = t$, then we can characterize the maximal Y–set of E ϕ–p.a.s. by the set of p–regular points of E.

2. Preliminaries

Let $\phi : [0,1] \to \mathbf{R}$ be a function which is increasing, continuous with $\phi(0) = 0$, $\phi(h) > 0$ for $h > 0$, and satisfies a smoothness condition. The smoothness condition is that there exists $c_\phi > 0$ such that $\phi(2s) \leq c_\phi \phi(s)$ for $0 \leq s \leq \frac{1}{2}$.

The Hausdorff measure of a set $E \subset \mathbf{R}^d$ is defined as

$$\phi - m(E) = \lim_{\delta \to 0} \inf\{\sum_{n=1}^{\infty} \phi(\text{ diam } G_n) : E \subset \cup_{n=1}^{\infty} G_n, \text{ diam } G_n \leq \delta\}$$

The $*$–Hausdorff measure of a set $E \subset \mathbf{R}^d$ (see [5]) is defined as $\phi - m^*(E) = \sup\{\phi - M^*(F) : F \subset E\}$, where

$$\phi - M^*(F) = \lim_{\delta \to 0} \inf\{\sum_{n=1}^{\infty} \phi(\text{ diam } B_n) : F \subset \cup_{n=1}^{\infty} B_n, \text{ diam } B_n \leq \delta,$$
$$B_n \text{ are open balls centered in } F\}$$

We note that $\phi - m(E) \leq \phi - m^*(E) \leq c\, \phi - m(E)$ for a suitable constant $c > 0$ [5].

We also note that an equivalent definition of $*$–Hausdorff measure is obtained if $\phi - m^*$ is defined for centered closed balls instead of centered open balls.

We recall the packing measure $\phi - p$ (see [9]) which is obtained by a two–stage definition using the pre–measure $\phi - \mathbf{P}$ defined for bounded sets

$E \subset \mathbf{R}^d$ as follows :

$$\phi - \mathbf{P}(E) = \lim_{\delta \to 0} \sup\{\sum_{n=1}^{\infty} \phi(\text{ diam } B_n) : B_n \text{ are disjoint open balls centered in } E \text{ with diam } B_n \leq \delta\}$$

It is immediate from the definition that $\phi - \mathbf{P}(E) = \phi - \mathbf{P}(\overline{E})$.

We employ Method I by Munroe [3] to obtain the outer measure :

$$\phi - p(E) = \inf\{\sum_{n=1}^{\infty} \phi - \mathbf{P}(E_n) : E_n \text{ are bounded }, E \subset \cup_{n=1}^{\infty} E_n\}$$

We note that $\phi - m$, $\phi - m^*$, and $\phi - p$ are metric outer measures ([1], [5], [8]) ; hence the corresponding classes of measurable sets include the Borel sets. Further, $\phi - m$ and $\phi - p$ are Borel regular and inner regular ([1], [8]). Also we see that if E is $\phi - p$ measurable, then E is $\phi - m$ measurable and $\phi - m^*$ measurable. Using the fact that $\phi - p$ is Borel regular, Borel sets are $\phi - m$ measurable and $\phi - m^*$ measurable, and $\phi - m \leq \phi - m^* \leq \phi - p$, we easily obtain the above result.

3. Density behaviour for a general measure μ

Throughout this section we assume that μ is a finite measure defined on the Borel subsets of $\mathbf{R}^d$. This implies that μ is inner regular.

If $B_r(x)$ denotes the closed ball centered at x with radius r, we define ϕ–densities by

$$\underline{D}_{\mu}{}^{\phi}(x) = \liminf_{r \downarrow 0} \mu(B_r(x))/\phi(2r)$$

$$\overline{D}_{\mu}{}^{\phi}(x) = \limsup_{r \downarrow 0} \mu(B_r(x))/\phi(2r)$$

We obtain equivalent ϕ–densities using $B_r(x)^0$ instead of $B_r(x)$. Now, we state four lemmas due to Raymond and Tricot (see Theorem 1.1 of [5]).

Lemma 3.1. *For any Borel set E with* $\phi - m^*(E) < \infty$,

$$\mu(E) \geq \phi - m^*(E) \inf\{\overline{D}_\mu{}^\phi(x) : x \in E\}.$$

Lemma 3.2. *For any Borel set E,*

$$\mu(E) \leq \phi - m^*(E) \sup\{\overline{D}_\mu{}^\phi(x) : x \in E\}.$$

Lemma 3.3. *For any Borel set E with* $\phi - p(E) < \infty$,

$$\mu(E) \geq \phi - p(E) \inf\{\underline{D}_\mu{}^\phi(x) : x \in E\}.$$

Lemma 3.4. *For any Borel set E,*

$$\mu(E) \leq \phi - p(E) \sup\{\underline{D}_\mu{}^\phi(x) : x \in E\}.$$

In the sequal, we will often apply the above lemmas to the measures $\mu(F) = \phi - m(E \cap F), \phi - m^*(E \cap F)$, and $\phi - p(E \cap F)$, so we introduce some notations for these cases. If $\phi - p(E) < \infty$ and $\mu(F) = \phi - p(E \cap F)$, put

$$\overline{D}_\mu{}^\phi(x) = \overline{\triangle}_\phi(x, E) = \limsup_{r \downarrow 0} \phi - p(B_r(x) \cap E)/\phi(2r)$$

$$\underline{D}_\mu{}^\phi(x) = \underline{\triangle}_\phi(x, E) = \liminf_{r \downarrow 0} \phi - p(B_r(x) \cap E)/\phi(2r)$$

Similarly, if $\phi - m(E) < \infty$ and $\mu(F) = \phi - m(E \cap F)$, put

$$\overline{D}_\mu{}^\phi(x) = \overline{D}_\phi(x, E) \text{ and } \underline{D}_\mu{}^\phi(x) = \underline{D}_\phi(x, E)$$

If $\phi - m^*(E) < \infty$ and $\mu(F) = \phi - m^*(E \cap F)$, in a similar manner, we can define $\overline{D}^*{}_\phi(x, E)$ and $\underline{D}^*{}_\phi(x, E)$.

If $\overline{\Delta}_\phi(x, E) = \underline{\Delta}_\phi(x, E)$, we write $\Delta_\phi(x, E)$ for the common value. Similarly, we write $D_\phi(x, E)$ and $D^*{}_\phi(x, E)$. In particular, when $0 < \phi - p(E) < \infty$ for $\phi - p$ measurable set E, a point $x \in E$ is called a p–regular point of E if $\underline{\Delta}_\phi(x, E) = \overline{\Delta}_\phi(x, E) = 1$; otherwise x is a p–irregular point. When $0 < \phi - p(E) < \infty$ for $\phi - p$ measurable set E, E is said to be p–regular if $\phi - p$–almost all of its points are p–regular, and p–irregular if almost all of its points are p–irregular. Similarly, we can define regularity and $*$–regularity for $\phi - m$ and $\phi - m^*$.

Next, we introduce two useful propositions which we shall require to prove the decomposition theorem for packing measures.

Proposition 3.5. *Suppose that E is $\phi - p$ measurable and $\phi - p(E) < \infty$. Then $\overline{\Delta}_\phi(x, E \backslash F) = 0$ ϕ-m.a.s. on F for $\phi - p$ measurable set $F \subset E$.*

Further, $\overline{\Delta}_\phi(x, F) = \overline{\Delta}_\phi(x, E)$ ϕ-m.a.s. on F.

Proof. See Corollary 7.4 of [4].

Proposition 3.6. *Let $G = \{x \in E : \overline{\Delta}_\phi(x, E) < k\}$, where E is $\phi - p$ measurable and $\phi - p(E) < \infty$, and k is a positive constant. For any $\phi - p$ measureble set $F \subset G$, if $\phi - m(F) = 0$, then $\phi - p(F) = 0$.*

Proof. As in the proof of Proposition 3.5, let E and F be Borel sets. Using Lemma 3.2 with $\mu(E) = \phi - p(E \cap F)$, we obtain $\phi - p(F) \leq \phi - m^*(F) \sup_{x \in F} \overline{\Delta}_\phi(x, E) \leq \phi - m^*(F)\, k \leq c_\phi k \phi - m(E)$. Hence, if $\phi - m(F) = 0$, then $\phi - p(F) = 0$.

Remark 3.7. It is easy to show that the above statement is true for

$G = \{x \in E : \overline{\triangle}_\phi(x, E) < \infty\}$; hence Corollary 4.6 of [9] is a special case of Proposition 3.6.

4. Main theorems

A subset $E \subset \mathbf{R}^d$ is said to be $*$–strongly ϕ–regular (strongly ϕ–regular) if E is $\phi - p$ measurable and $0 < \phi - m^*(E) = \phi - p(E) < \infty$ (if E is $\phi - p$ measurable and $0 < \phi - m(E) = \phi - p(E) < \infty$).

We list the next six lemmas essentially due to Raymond and Tricot (see Corollaries 7.1, 7.2, and 8.1, Propositions 9.1 and 9.2, and Corollary 9.5 of [5]).

Lemma 4.1. *If E is $\phi - p$ measurable and $\phi - p(E) < \infty$, then $\underline{\triangle}_\phi(x, E) = 1$ ϕ-p.a.s. on E.*

Lemma 4.2. *If E is $\phi - p$ measurable and $\phi - p(E) < \infty$, then $\overline{D}^*{}_\phi(x, E) = 1$ ϕ-m^*.a.s. (ϕ–m.a.s.) on E.*

Lemma 4.3. *If E is $\phi - p$ measurable and $0 < \phi - p(E) < \infty$, then $\phi - m(E) = 0$ if and only if $\overline{\triangle}_\phi(x, E) = \infty$ ϕ–p.a.s. on E.*

Lemma 4.4. *If E is $\phi - p$ measurable and $\phi - p(E) < \infty$, the following statements are equivalent :*

1) $\phi - m^*(E) = \phi - p(E)$.

2) $\overline{\triangle}_\phi(x, E) = 1$ *ϕ–p.a.s. on E.*

3) $\triangle_\phi(x, E) = 1$ *ϕ–p.a.s. on E.*

Lemma 4.5. *If E is $\phi - p$ measurable and $\phi - p(E) < \infty$, the following statements are equivalent :*

1) $\phi - m(E) = \phi - p(E)$.

2) $D_\phi(x,E) = 1$ *ϕ–p.a.s. on E.*

Lemma 4.6. *Let $\phi(t) = t^k$ where $k \in \mathbf{N}$. If E is $\phi - p$ measurable and $\phi - p(E) < \infty$, the following statements are equivalent :*

1) $\phi - m(E) = \phi - p(E)$.

2) $\phi - m^*(E) = \phi - p(E)$.

The proofs of the next two lemmas are similar to those of Theorem 6.2 and Corollary 6.3 of [9], with the use of Lemmas 3.3 and 3.4.

Lemma 4.7. *If E is $\phi - p$ measurable and $\phi - p(E) < \infty$, then E is $*$-strongly ϕ-regular if and only if $\underline{D}^*{}_\phi(x,E) = 1$ ϕ–p.a.s. on E.*

Lemma 4.8. *If E is ϕ–p measurable, ϕ–$p(E) < \infty$, and $\overline{\triangle}_\phi(x,E) < \infty$ ϕ–p.a.s. on E, then E is $*$-strongly ϕ-regular if and only if it is $*$-regular.*

Now we state a decomposition theorem of Besicovitch type for packing measures.

Theorem 4.9. (Decomposition theorem) *If E is $\phi - p$ measurable and $\phi - p(E) < \infty$, then the set of p–regular points of E is a p-regular set, and the set of p-irregular points of E is a p–irregular set.*

Proof. First, by lemma 4.1, we only need to show that $\overline{\triangle}_\phi(x,\triangle) = 1$ ϕ–p.a.s. on $\triangle$, where $\triangle = \{x \in E : \triangle_\phi(x,E) = 1\}$.

By Proposition 3.5, $\overline{\triangle}_\phi(x,\triangle) = \overline{\triangle}_\phi(x,E)$ ϕ–m.a.s. on $\triangle$.

By Proposition 3,6, $\overline{\triangle}_\phi(x,\triangle) = \overline{\triangle}_\phi(x,E)$ ϕ–p.a.s. on $\triangle$.

Second, we must show that it is not ϕ–p.a.s. on $E\backslash\triangle$ that $\triangle_\phi(x,E\backslash\triangle) = 1$.

By Proposition 3.5, on $E\backslash\Delta$ it is not ϕ–m.a.s. that $\Delta_\phi(x, E\backslash\Delta) = 1$. Therefore, by Proposition 3.6 that $\phi - p(\{x \in E\backslash\Delta : \Delta_\phi(x, E\backslash\Delta) = 1\}) = 0$.

Theorem 4.10 (Propositions 11.1 and 11.2 of [4]) *Let E be any set in $\mathbf{R}^d$. If $\phi - m^*(E) = \phi - p(E) < \infty$, then $\phi - m^*(A) = \phi - p(A)$ for any $\phi - m^*$ measurable set $A \subset E$. Further, if $\phi - m(E) = \phi - p(E) < \infty$, then $\phi - m(A) = \phi - p(A)$ for any set $A \subset E$.*

Proof. It is immediate from the fact that $\phi - m$ and $\phi - p$ are Borel regular.

We remark that, if $\phi - m^*$ measurable $A \subset \Delta$ ϕ–p.a.s., then $\phi - m^*(A) = \phi - p(A)$. The following theorem is the converse of this remark.

Theorem 4.11. *Let E be $\phi - p$ measurable and $\phi - p(E) < \infty$. If $\phi - m^*(A) = \phi - p(A)$, where $A \subset E$, then $A \subset \{x \in E : \Delta_\phi(x, E) = 1\}$ ϕ-p.a.s.*

Proof. We may assume that A is a Borel set. Suppose that $\phi - p(A\backslash\Delta) > 0$, where $\Delta = \{x \in E : \Delta_\phi(x, E) = 1\}$.

Then $\phi - p(A\backslash\Delta) = \phi - m^*(A\backslash\Delta)$ by Theorem 4.10, and $\overline{\Delta}_\phi(x, A\backslash\Delta) = 1$ ϕ–p.a.s. on $A\backslash\Delta$ by Lemma 4.4. By Proposition 3.5, $\overline{\Delta}_\phi(x, E) = 1$ ϕ–m.a.s. on $A\backslash\Delta$. Combining this with Lemma 4.1, we obtain that $\Delta_\phi(x, E) = 1$ ϕ–m.a.s. on $A\backslash\Delta$. Proposition 3.6 then yields $\Delta_\phi(x, E) = 1$ ϕ–p.a.s. on $A\backslash\Delta$, which is a contradiction.

Remark 4.12. If $\phi - p(A) > 0$ for a subset A of a p–irregular set, then $\phi - m^*(A) < \phi - p(A)$.

Theorem 4.13. *If E is $\phi - p$ measurable and $\phi - p(E) < \infty$, then $\{x \in E : \Delta_\phi(x, E) = 1\} \subset \{x \in E : D^*{}_\phi(x, E) = 1\}$ ϕ-p.a.s..(i.e., the set of* *p-regular points of E is contained in the set of ∗-regular points of E ϕ–p.a.s.)*

Proof. We may assume that E is a Borel set. Let $\Delta = \{x \in E : \Delta_\phi(x, E) = 1\}$ and $D^* = \{x \in E : {D^*}_\phi(x, E) = 1\}$

By the decomposition theorem (Theorem 4.9), Δ is a p–regular set ; hence ∗–strongly ϕ–regular by Lemma 4.4. Since Δ is $\phi - p$ measurable, we have $\underline{D^*}_\phi(x, \Delta) = 1$ ϕ–p.a.s. on Δ, by Lemma 4.7. Thus, $\underline{D^*}_\phi(x, E) \geq 1$ ϕ–p.a.s. on Δ. Together with Lemma 4.2 and Proposition 3.6, we obtain that ${D^*}_\phi(x, E) = 1$ ϕ–p.a.s. on Δ.

Theorem 4.14. *If E is $\phi - p$ measurable and $\phi - p(E) < \infty$, then $\phi - m(D^* \backslash \Delta) = 0$, where D^* is the set of ∗-regular points of E, and Δ is the set of p-regular points of E.*

Proof. We may assume that E is a Borel set. Let $X = D^* \backslash \Delta$. Then X is ∗–regular, since D^* is ∗–regular and Δ is a Borel set. Suppose that $\phi - m(X) > 0$. Then X is not ∗–strongly ϕ–regular by Remark 4.12. By Lemma 4.8, it is not true that $\overline{\Delta}_\phi(x, X) < \infty$ ϕ–p.a.s on X. In fact, $\overline{\Delta}_\phi(x, X) = \infty$, ϕ–p.a.s., on X using again Lemma 4.8.. For, let $X_1 = \{x \in X : \overline{\Delta}_\phi(x, X) < \infty\}$. Then $\phi - p(X_1) = 0$, since X_1 is ∗–regular and $\overline{\Delta}_\phi(x, X_1) \leq \overline{\Delta}_\phi(x, X) < \infty$ on X_1. By Lemma 4.3, $\phi - m(X) = 0$. This is a contradiction.

Corollary 4.15. *If E is $\phi - p$ measurable and $\phi - p(E) < \infty$, then $\phi - m(D \backslash \Delta) = 0$, where D is the set of regular points of E, and Δ is the set of p-regular points of E.*

Proof. Clearly, D is a subset of D^* ϕ–m.a.s.. From Theorem 4.13 and Theorem 4.14 follows our result.

Theorem 4.16. *Let $\phi(t) = t^k$, where $k \in \mathbf{N}$. If E is $\phi - p$ measurable with $\phi - p(E) < \infty$, then $\Delta \subset D$ ϕ-p.a.s. and $D = D^*$ ϕ-m.a.s., where*

Δ, D, and D^* are the set of p-regular, regular, and $*$–regular points of E respectively.

Proof. First, Δ is p–regular; hence $\phi - m(\Delta) = \phi - p(\Delta)$, by Lemma 4.4 and Lemma 4.6. By Lemma 4.5, Δ is regular. Thus $D_\phi(x, \Delta) = 1$ ϕ–m.a.s. on Δ. Since $D_\phi(x, E\backslash\Delta) = 0$ on Δ (Corollary 2.4 of [1]), $D_\phi(x, E) = 1$ ϕ–m.a.s. on Δ. By Proposition 3.6, it follows that $D_\phi(x, E) = 1$ ϕ–p.a.s. on Δ. That is, $\Delta \subset D\phi$–p.a.s..

Second, trivially $D \subset D^*$ ϕ–m.a.s.. Noting that $\phi - m(D^*\backslash\Delta) = 0$ by Theorem 4.14 and $\Delta \subset D$ ϕ–m.a.s., we conclude that $\phi - m(D^*\backslash D) = 0$.

We define E to be a Y–set if it is included in a countable union of rectifiable arcs [9].

Theorem 4.17. *If E is $\phi - p$ measurable and $\phi - p(E) < \infty$ for $\phi(t) = t$, then Δ, the set of p-regular points of E, is the maximal Y-set of E ϕ-p.a.s.*

Proof. Since Δ is $\phi - p$ measurable, Δ is $\phi - m$ measurable. By Corollary 6.4 of [9], $\Delta = \Delta_1 \cup \Delta_2$, where Δ_1 is a Y–set and $\phi - p(\Delta_2) = 0$. We easily show that Δ_1 is strongly regular.

By Remark 4.12, we see that there is no strongly regular set A such that $A \subset E\backslash\Delta$ and $\phi - p(A) > 0$.

Hence Δ is the maximal Y–set of E ϕ–p.a.s.

REFERENCES

[1] K.J. Falconer, *The Geometry of Fractal Sets* (Cambridge University Press, 1985).

[2] M.de Guzmán, *Differentiation of Integrals in* $\mathbf{R}^n$. (Springer Lecture

Note in Mathematics 481, 1975).

[3] M. Munroe, *Measure and Integration* (Addison–Wesley, 1971).

[4] X.S. Raymond and C. Tricot, Packing regularity of sets in N-space, *Centre de Recherches Mathematiques–1413, Octobre 1986, Université de Montreal.*

[5] X.S. Raymond and C. Tricot, Packing regularity of sets in n-space, *Math. Proc. Camb. Phil. Soc.* 103(1988), 133–145.

[6] C.A. Rogers, *The Hausdorff measures* (Cambridge University Press, 1970).

[7] S.J. Taylor and C. Tricot, The packing measure of rectifiable sets, *Real Analysis Exchange*, Vol.10, No. 1 (1984–85), 58–67.

[8] S.J. Taylor and C. Tricot, Packing measure, and it evaluation for a Brownian path. *Trans. Amer. Math. Soc.* 288(1985), 679–699.

[9] S.J. Taylor and C. Tricot, The packing measure of recitifiable subsets of the plane, *Math. Proc. Camb. Phil. Soc.*, 99(1986), 285–296.

[10] C. Tricot, Two definitions of fractional dimension, *Math. Proc. Camb. Phil. Soc.*, 91(1982), 57–74.

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