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Measures With Prescribed Marginals, Extreme Points and Measure Preserving Transformations

Let $(X, \mathcal{A}, \lambda)$ and $(Y, \mathcal{B}, \nu)$ be two probability spaces. Let $M(\lambda, \nu)$ be the collection of all probability measures μ on the product σ-field $\mathcal{A} \times \mathcal{B}$ of $X \times Y$ such that the first and second marginals of μ are λ and ν, respectively, i.e., $\mu_1(A) = \mu(A \times Y) = \lambda(A)$ for every A in $\mathcal{A}$, and $\mu_2(B) = \mu(X \times B) = \nu(B)$ for every B in $\mathcal{B}$. The set $M(\lambda, \nu)$ is convex. The extreme points of this set have been characterized by Douglas (1964, Theorem 1, p.243) and Lindenstrauss (1965) when $X = Y$, $\mathcal{A} = \mathcal{B}$, $\lambda = \nu$ and the probability space has some additional structure. Let T be a measure preserving transformation from X to Y, i.e., T is a measurable transformation from X to Y, and $\lambda(T^{-1}B) = \nu(B)$ for every B in $\mathcal{B}$. We show that every such transformation gives an extreme point of $M(\lambda, \nu)$. The basic idea is to build a probability measure μ_T in $M(\lambda, \nu)$ sitting on the graph $G = \{(x, Tx);\ x \in X\}$ of T. But the graph G of T need not be available in the product σ-field $\mathcal{A} \times \mathcal{B}$. See Rao and Rao (1981, p.17) or Rao (1969). We overcome this difficulty by proceeding as follows and obtain a measure μ_T for which G is a thick set.

Let P_1 be the projection map from $X \times Y$ to X. We claim that the graph G has the property: for every E in $\mathcal{A} \times \mathcal{B}$, $P_1(E \cap G) \in \mathcal{A}$. For, let $\mathcal{E} = \{E \in \mathcal{A} \times \mathcal{B};\ P_1(E \cap G) \in \mathcal{A}\}$. One can show that $\mathcal{E}$ is closed under complementation and countable unions, and contains all measurable rectangle sets. Hence $\mathcal{E} = \mathcal{A} \times \mathcal{B}$. Define a set function μ_T on $\mathcal{A} \times \mathcal{B}$ by

$$\mu_T(E) = \lambda(P_1(E \cap G)) \text{ for } E \text{ in } \mathcal{A} \times \mathcal{B}.$$

<u>**THEOREM.**</u> μ_T is an extreme point of $M(\lambda, \nu)$.

<u>**Proof.**</u> It is easy to check that μ_T is a probability measure on $\mathcal{A} \times \mathcal{B}$. We now check that μ_T has the prescribed marginals. Let $A \in \mathcal{A}$. Then $\mu_1(A) = \mu_T(A \times Y) = \lambda[P_1((A \times Y) \cap G))] = \lambda(A \cap T^{-1}Y) = \lambda(A)$. Let $B \in \mathcal{B}$. Then

$\mu_2(B) = \mu_T(X \times B) = \lambda[P_1((X \times B) \cap G)] = \lambda(X \cap T^{-1}B) = \lambda(T^{-1}B) = \nu(B)$, since T is measure preserving. We now claim that G is a thick subset of $X \times Y$ under μ_T, i.e., the outermeasure of G, $\mu_T^*(G) = 1$. For, if E is any set in $\mathcal{A} \times \mathcal{B}$ containing G, then $P_1(E \cap G) = X$. Finally, we assert that μ_T is an extreme point of $M(\lambda, \nu)$. Suppose $\mu_T = (1/2)(\zeta + \eta)$ for some ζ and η in $M(\lambda, \nu)$. It is obvious that $\zeta^*(G) = \eta^*(G) = 1$. Further, for any E in $\mathcal{A} \times \mathcal{B}$, $\zeta^*(E \cap G) = \zeta(E)$ and $\eta^*(E \cap G) = \eta(E)$. See Halmos (1950, Theorem A, p.75). If $A \times B \in \mathcal{A} \times \mathcal{B}$, then $(A \times B) \cap G \subset (A \cap T^{-1}B) \times Y$. Consequently,

$$\zeta(A \times B) \leq \zeta[(A \cap T^{-1}B) \times Y] = \lambda(A \cap T^{-1}B) = \mu_T(A \times B),$$

and

$$\eta(A \times B) \leq \eta[(A \cap T^{-1}B) \times Y] = \lambda(A \cap T^{-1}B) = \mu_T(A \times B).$$

Hence $\mu_T(A \times B) = \zeta(A \times B) = \eta(A \times B)$ for every A in $\mathcal{A}$ and B in $\mathcal{B}$. Therefore, $\mu_T = \zeta = \eta$. This completes the proof.

Remarks. 1. Brown (1966, p.17-19) proved the above result when $X = Y$, $\lambda = \nu$, and the probability space $(X, \mathcal{A}, \lambda)$ is homogeneous and nonatomic.

2. There are cases that every extreme point of $M(\lambda, \nu)$ comes from some measure preserving transformation. As an example, let $X = Y = \{1, 2, 3\}$, $\lambda(\{1\}) = \lambda(\{2\}) = \lambda(\{3\}) = 1/3$ and $\nu(\{1\}) = \nu(\{2\}) = \nu(\{3\}) = 1/3$. The set $M(\lambda, \nu)$ can be identified as the collection of all doubly stochastic matrices of order 3×3 with each row and column sum equal to $1/3$. By the well-known Birkhoff's theorem, the extreme points of $M(\lambda, \nu)$ can be identified with the six permutation matrices of order 3×3. Every one-to-one and onto transformation from X to Y is measure preserving. All these six transformations give all the extreme points of $M(\lambda, \nu)$.

3. There are cases that only some extreme points of $M(\lambda, \nu)$ come from measure preserving transformations. As an example, let $X = Y = \{1, 2, 3\}$, $\lambda(\{1\}) = 1/6$, $\lambda(\{2\}) = 1/3$, $\lambda(\{3\}) = 1/2$, $\nu(\{1\}) = 1/6$, $\nu(\{2\}) = 1/3$, and $\nu(\{3\}) = 1/2$. The only measure preserving transformation in this case is the identity transformation. Surely, there are more extreme points of $M(\lambda, \nu)$.

4. There are cases in which no measure preserving transformation exists. As an example, let $X = Y = \{1, 2, 3\}$, $\lambda(\{1\}) = \lambda(\{2\}) = \lambda(\{3\}) = 1/3$, $\nu(\{1\}) = 1/8$, $\nu(\{2\}) = 2/8$, and $\nu(\{3\}) = 5/8$.

5. Let $X = Y = [0, 1]$, $\mathcal{A} = \mathcal{B} =$ Lebesgue σ-field on X, and $\lambda = \nu =$ Lebesgue measure on $\mathcal{A}$. Then there are uncountably many measure preserving transformations on X preserving the Lebesgue measure. Further, if T_1 and T_2 are two distinct invertible measure preserving transformations modulo a null set, i.e., $\lambda(x \in X;\ T_1 x = T_2 x) \neq 1$, then μ_{T_1} and μ_{T_2} are distinct. For, if $\mu_{T_1} = \mu_{T_2}$, then $\mu_{T_1}(A \times B) = \mu_{T_2}(A \times B) = \lambda(A \cap T_1^{-1}B) = \lambda(A \cap T_2^{-1}B)$ for every A and B in $\mathcal{B}$. This implies that $\lambda(T_1^{-1}B \,\Delta\, T_2^{-1}B) = 0$ for every B in $\mathcal{B}$. Consequently, as set transformations from $\mathcal{B}$(modulo λ-null sets) to $\mathcal{B}$(modulo null sets), T_1^{-1} and T_2^{-1} are identical. Hence $T_1 = T_2$ a.e. $[\lambda]$. Not every extreme point of $M(\lambda, \lambda)$ comes from an invertible measure preserving transformation. An example can be given. This is in contrast to the case when $X = Y$, X is a finite set, $\lambda = \nu$, λ is the uniform probability measure on X in which every extreme point of $M(\lambda, \lambda)$ comes from a measure preserving transformation.

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References

[1] J.R. Brown, Approximation Theorems for Markov Operators, Pacific J. Math., 16 (1966), 13-23.

[2] R.G. Douglas, On Extremal Measures and Subspace Density, Michigan Math. J., 11 (1964), 242-246.

[3] P.R. Halmos, Measure Theory, Van Nostrand, New York, 1950.

[4] J. Lindenstrauss, A Remark on Extreme Doubly Stochastic Measures, Amer. Math. Monthly, 72 (1965), 379-382.

[5] B.V. Rao, On Discrete Borel Spaces and Projective Sets, Bull. Amer. Math. Soc., 75 (1969), 614-617.

[6] K.P.S. Bhaskara Rao and B.V. Rao, Borel Spaces, Dissertationes Mathematicae, 190 (1981), Panstwowe Wydawnictwo Naukowe, Warszawa.

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