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ON UNIVERSALLY BAD DARBOUX FUNCTIONS

Abstract

It is well known that the sum (and the product) of a continuous function and a Darboux function need not be Darboux in general ([9]). More precisely, for every nowhere constant continuous $g : \mathbf{R} \to \mathbf{R}$ there exists some "bad" Darboux function $f : \mathbf{R} \to \mathbf{R}$ such that $f + g$ or $f \cdot g$ do not have the Darboux property, see [2], [8]. It is the purpose of the present paper to construct a "universally bad" Darboux function f, see Corollary 2 below.

I

Let us establish some of the notation to be used later. We shall be concerned with real-valued functions defined on a subinterval I of $\mathbf{R}$; here all intervals are assumed to be nondegenerate. Such a function f is said to be a Darboux function if $f(J)$ is connected for any interval $J \subset I$. For a set $A \subset \mathbf{R}$ we denote by $\mathcal{D}^*(I, A)$ the set of all $f : I \to A$ such that $\mathbf{cl}\ f^{-1}(y) = I$ for any $y \in A$ and we set $\mathcal{D}^* = \mathcal{D}^*(\mathbf{R}, \mathbf{R})$. A function f is said to be nowhere constant if no one of its level sets $f^{-1}(y)$ contains a relatively open subset of dom f. If G is a real-valued function defined on a subset of the plane, then we define for any $t \in \mathbf{R}$ the horizontal section of G by $G^t(x) = G(x, t)$ whenever $(x, t) \in$ dom G, the definition of the vertical section G_t is analogous.

In the proof of Theorem 1 below we shall use the following

Assumption (**A**): the union of fewer than $\mathfrak{c}$ (power of continuum) first-category subsets of $\mathbf{R}$ is again of the first category.

The logical status of (**A**) perhaps requires some explanation. First of all, note that assumption (**A**) is independent of **ZFC**. Indeed, already in Cohen's classical model for (**ZFC**) + non **CH**), see [5], there are subsets of the real line[1] which are not of the first category but whose cardinalities are less than $\mathfrak{c}$. A proof of this statement - formulated in a different but equivalent language - can be found in [4].

Next, (**A**) is a widely used consequence of Martin's axiom. (See the quite popular paper [10] for basic informations.) Since Martin's axiom is strictly weaker than the continuum hypothesis ([7]), we infer that **CH** implies (**A**) but not conversely. But even Martin's axiom is not implied by (**A**) since for instance **ZFC** + (**A**) + "$\mathbf{R}$ is the union of fewer than $\mathfrak{c}$ Lebesgue zero-sets" is consistent. However, Martin's axiom is equivalent to the assumption that any nonvoid compact Hausdorff space not containing an uncountable collection of open sets cannot be written as the union of fewer than $\mathfrak{c}$ first category sets. For both of these facts as well as for a rather extensive treatment of Martin's axiom we refer the reader to [6], in particular Theorems B1H, B1G, and 13A.

1. Theorem *Let D be a dense, second category subset of $\mathbf{R}$. Then there is a function $f \in \mathcal{D}^*(\mathbf{R}, D)$ such that the function*

$$x \to G(f(x), g(x))$$

does not have the Darboux property on the nondegenerate interval $I \subset \mathbf{R}$ whenever

$$g : I \to \mathbf{R} \text{ is continuous and nowhere constant.} \tag{1}$$

$$G : \mathbf{R} \times g(I) \to \mathbf{R} \text{ is continuous,} \tag{2}$$

and

$$\left.\begin{array}{r}\text{all, except countably many of the sections}\\ G_t \text{ and } G^t,\ t \in \mathbf{R}, \text{ are nowhere constant.}\end{array}\right\} \tag{3}$$

Proof Let $\mathcal{M}$ be the family of all triples (G, g, I) fulfilling (1), (2), and (3) above. Then the cardinality of $\mathcal{M}$ is less than or equal to $\mathfrak{c}$. Hence, we can write

$$\mathcal{M} = \{(G_\alpha, g_\alpha, I_\alpha); \alpha < \mathfrak{c}\}.$$

[1] take e.g. the set of all constructible real numbers

First, notice that for any function h defined on an interval, there are at most countably many y such that $h^{-1}(y)$ contains an interval. This together with (3) implies that for any $\alpha < \mathfrak{c}$ the set

$$C_\alpha = \{y;\ \mathbf{int}[(G_\alpha)^t]^{-1}(y) \neq \phi \text{ or } \mathbf{int}[(G_\alpha)_t]^{-1}(y) \neq \phi \text{ for some } t\}$$

is countable. In the sequel we will use the fact that for any $y \notin C_\alpha$ and for any $t \in \mathbf{R}$ both sets $\{x; G_\alpha(x,t) = y\}$ and $\{x; G_\alpha(t,x) = y\}$ are nowhere dense in $\mathbf{R}$.

Further, let $\{U_i; i < \omega\}$ be a sequence of intervals forming a base for the euclidean topology in $\mathbf{R}$ and $\mathbf{R} = \{x_\alpha; \alpha < \mathfrak{c}\}$.

We will inductively define sequences of points

$$t_{\alpha,i} \in U_i, y_\alpha \in \mathbf{R}, p_\alpha \in g_\alpha(I_\alpha), \text{ and } z_\alpha \in D \text{ for } \alpha < \mathfrak{c} \text{ and } i < \omega \tag{4}$$

such that

$$\begin{aligned}
t_{\alpha,i} &= t_{\beta,j} \text{ implies } \alpha = \beta \text{ and } i = j && (5)\\
t_{\beta,i} &\in I_\alpha \text{ implies } G_\alpha(x_\beta, g_\alpha(t_{\beta,i})) \neq y_\alpha \text{ for } \alpha, \beta < \mathfrak{c} \text{ and } i < \omega && (6)\\
x_\beta &\in I_\alpha \text{ implies } G_\alpha(z_\beta, g_\alpha(x_\beta)) \neq y_\alpha \text{ for } \alpha, \beta < \mathfrak{c} && (7)\\
y_\alpha &\notin C_\alpha \text{ for } \alpha < \mathfrak{c}, \text{ and} && (8)\\
y_\alpha &\in (\inf G_\alpha(\mathbf{R} \times \{p_\alpha\}), \sup G_\alpha(\mathbf{R} \times \{p_\alpha\})) \text{ for } \alpha < \mathfrak{c}. && (9)
\end{aligned}$$

For this purpose, let us assume that for some $\alpha < \mathfrak{c}$ all $t_{\beta,i}, y_\beta, z_\beta$ and p_β with $\beta < \alpha$ and $i < \omega$ are already defined. According to (8) and (1), for any $\beta < \alpha$ both sets $\{x; G_\beta(x, g_\beta(x_\alpha)) = y_\beta\}$ and $\{x \in I_\alpha; G_\beta(x_\alpha, g_\beta(x)) = y_\beta\}$ are nowhere dense. Hence, assumption **(A)** implies the existence of

$$z_\alpha \in D \setminus \bigcup_{\beta<\alpha} \{x; G_\beta(x, g_\beta(x_\alpha)) = y_\beta\} \tag{10}$$

as well as of

$$\begin{aligned}
t_{\alpha,i} \in U_i \setminus (\cup_{\beta<\alpha}\{x &\in I_\beta; G_\beta(x_\alpha, g_\beta(x)) = y_\beta\}\\
&\cup \{t_{\beta,j}; \beta < \alpha \text{ or } (\beta = \alpha \text{ and } j < i)\})
\end{aligned} \tag{11}$$

for $i = 1, 2, \ldots$. Next, we select any $p_\alpha \in g_\alpha(I_\alpha)$ such that $(G_\alpha)^{p_\alpha}$ is nonconstant. Since the set K_α which is defined to be

$$\bigcup_{\beta\leq\alpha} (\bigcup_{i<\omega} \{G_\alpha(x_\beta, g_\alpha(t_{\beta,i})); t_{\beta,i} \in I_\alpha\} \cup \{G_\alpha(z_\beta, g_\alpha(x_\beta)); x_\beta \in I_\alpha\} \cup C_\beta) \tag{12}$$

is of cardinality less than $\mathfrak{c}$, we can choose some

$$y_\alpha \in (\inf G_\alpha(\mathbf{R} \times \{p_\alpha\}),\ \sup G_\alpha(\mathbf{R} \times \{p_\alpha\}))\setminus K_\alpha. \tag{13}$$

Evidently, the $t_{\alpha,i}$'s, y_α's, z_α's and p_α's chosen in this way satisfy (4), (5), (8), and (9). To verify (6) fix any $i < \omega$, $\alpha, \beta < \mathfrak{c}$ and let $t_{\beta,i} \in I_\alpha$. If $\beta \leq \alpha$, then (12) and (13) together ensure that $G_\alpha(x_\beta, g_\alpha(t_{\beta,i})) \neq y_\alpha$. Else we have $\beta > \alpha$ and in this case (11) implies (6). Similarly (7) can be shown.

Now we define the desired function f by

$$f(x) = \begin{cases} x_\alpha & \text{if } x = t_{\alpha,i} \text{ and } x_\alpha \in D \text{ for some } i < \omega \text{ and } \alpha < \mathfrak{c} \\ z_\alpha & \text{if } x_\alpha = x \notin \{t_{\beta,i};\ x_\beta \in D,\ i < \omega \text{ and } \beta < \mathfrak{c}\} \end{cases}$$

Since for each $x \in D$ there is some $x_\alpha = x$ and since $f(t_{\alpha,i}) = x$ for any $i < \omega$, we conclude from $t_{\alpha,i} \in U_i$ that $f \in \mathcal{D}^*(\mathbf{R}, D)$.

We finish the proof by showing that for any fixed $\alpha < \mathfrak{c}$ the function $h(x) = G_\alpha(f(x), g_\alpha(x))$, $x \in I_\alpha$ fulfills

$$y_\alpha \notin h(I_\alpha) \text{ but } \inf h(I_\alpha) < y_\alpha < \sup h(I_\alpha).$$

Indeed, the first statement holds since in case $x = t_{\beta,i} \in I_\alpha$ with $x_\beta \in D$ (6) implies $h(x) = G_\alpha(x_\beta, g_\alpha(t_{\beta,i})) \neq y_\alpha$ and since for other $x = x_\beta \in I_\alpha$ $h(x) = G_\alpha(z_\beta, g_\alpha(x_\beta)) \neq y_\alpha$ by (7). As concerns the second statement, we first conclude from (9), $\mathbf{cl}\, D = \mathbf{R}$, and (2) that there are $v, w \in D$ satisfying

$$G_\alpha(v, p_\alpha) < y_\alpha < G_\alpha(w, p_\alpha).$$

Since $f \in \mathcal{D}^*(\mathbf{R}, D)$ and $p_\alpha = g_\alpha(s)$ for some $s \in I_\alpha$, there exist sequences $\{a_i\}_{i=1}^\infty$, $\{b_i\}_{i=1}^\infty$ of points from I_α such that $\lim_{i\to\infty} a_i = \lim_{i\to\infty} b_i = s$, $f(a_i) = v$, and $f(b_i) = w$ for $i \geq 1$. Then (1) and (2) yields $\lim_{i\to\infty} h(a_i) = \lim_{i\to\infty} G_\alpha(v, g_\alpha(a_i)) = G_\alpha(v, p_\alpha) < y_\alpha$ and similarly $\lim_{i\to\infty} h(b_i) > y_\alpha$ which of course implies the second statement and finishes the proof.

We want to point out the most interesting case of our quite general theorem.

2. Corollary *There is an $f \in \mathcal{D}^*$ such that for each continuous and nowhere constant function g defined on some interval the functions $f + g, f - g, f \cdot g$ and f/g (if $0 \notin$ **rng**(g)) do not have the Darboux property.*

3. Remark It seems not to be very easy to get rid of the condition that g is nowhere constant. Indeed, if $f \in \mathcal{D}^*$ and g is a continuous Cantor-type function, i.e. the set of points of local constantcy of g is dense in dom g, then obviously $f + g \in \mathcal{D}^*$ has the Darboux property. This motivates us to formulate

Problem 1. Does there exist a Darboux function $f : \mathbf{R} \to \mathbf{R}$ such that $f + g(f \cdot g)$ does not have the Darboux property whenever the continuous function g is nonconstant?

Problem 2. Is there a Darboux function $f : \mathbf{R} \to \mathbf{R}$ such that the points of continuity form a dense set and that the function $x \to G(f(x), g(x))$ is not Darboux on the interval $I \subset \mathbf{R}$ whenever (1), (2), and (3) from Theorem 1 are fulfilled[2]. (It is not very difficult to see that no such function can serve as a solution of Problem 1).

II

Let $I \subset \mathbf{R}$ be a interval. In [3] it was stated (in a more general form) that for any Darboux $f : I \to \mathbf{R}$ and any continuous $g : I \to \mathbf{R}$ the sum $f + g$ belongs to the class $\mathcal{U}(I)$ of all $h : I \to \mathbf{R}$ such that $\mathbf{cl}\ f(J \backslash A) = [\inf f(J),\ \sup f(J)]$ whenever J is a subinterval of I and the cardinality of A is less than $\mathfrak{c}$. On the other hand, our theorem shows that under the assumption (**A**) there is always a function in $\mathcal{U}(I)$ which cannot be written as the sum of a Darboux and a continuous function. Indeed, notice that for $D = \mathbf{R} \backslash \mathbf{Q}$, the irrationals, $\mathcal{D}^*(I, D) \subset \mathcal{U}(I)$ holds. Let $F \in \mathcal{D}^*(I, D)$ be the restriction (to I) of the function f from Theorem 1 and let $F = d + g$, where $g : I \to \mathbf{R}$ is continuous. If g is constant on some interval $J \subset I$, then $d = F - g$ is not Darboux on J and if g is nowhere constant then according to Theorem 1 $d = F + (-g)$ again does not have the Darboux property. Therefore, we are led to

Problem 3. Characterize the class of all functions which are the sum of a Darboux and a continuous function.

However, this question appeared already a long ago - see the survey paper [1].

References

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