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A BERNSTEIN PARTING OF A SPACE OF MEASURABLE SETS

Let E be a Lebesgue measurable subset of the real line R such that $0 < m(E) < \infty$. Let X denote the family of measurable subsets of E modulo sets of measure zero; that is, subsets A and B of E are the same element in X if $m(A \setminus B) = m(B \setminus A) = 0$. For sets B and C in X, let the distance between B and C be $m(B \setminus C) + m(C \setminus B)$. The resulting metric space X is separable. (Consider all sets of the form $U \cap E$ where U is the union of finitely many intervals with rational endpoints.)

Definition. By a Bernstein parting of a separable metric space Y, we mean a partition of Y into two disjoint connected dense subsets.

It is known [SS] that the plane R^2 has a Bernstein parting. In this note we prove that certain other separable metric spaces have Bernstein partings. In particular, the metric space X defined above has a Bernstein parting. It will be easier to prove that the Hilbert cube (see [HS]), and any separable metric linear space of dimension ≥ 2, have Bernstein partings (see Lemma 1). On the other hand, the circle in R^2 and the real line R obviously do not have Bernstein partings. Any separable metric space that has a Bernstein parting is necessarily connected.

Lemma 1. Let Y be a separable metric space such that for any points a and b in Y, there is a subspace P of Y containing a and b such that P is homeomorphic to the closed unit disc in R^2. Then Y has a Bernstein parting.

Proof. Let Γ denote the first ordinal number with cardinality of the continuum, c. Because Y is separable, there are at most c closed connected nondegenerate subsets of Y. Use the ordinal numbers $\alpha < \Gamma$ to index the closed connected nondegenerate subsets of Y, $\{C_\alpha\}_{\alpha<\Gamma}$. Now each C_α has at least c points because C_α is connected and Y is a metric space. (If $t, s \in C_\alpha$ then any ball with center t that does not contain s has a boundary that meets C_α.) By transfinite induction we choose two distinct elements a_α and b_α in C_α ($\alpha < \Gamma$) such that the sets $A = \bigcup_{\alpha<\Gamma}\{a_\alpha\}$ and $B = \bigcup_{\alpha<\Gamma}\{b_\alpha\}$ are disjoint.

We claim that A and B are dense in Y. Let $y \in U$ and U be an open subset of Y. By hypothesis there is a subspace P_1 of Y such that $y \in P_1$ and P_1 is homeomorphic to the closed unit disc in R^2. Then for some $\alpha < \Gamma$, $C_\alpha \subset P_1 \cap U$ because any open subset of R^2 contains a closed connected nondegenerate subset. Thus $a_\alpha \in P_1 \cap U$ and $b_\alpha \in P_1 \cap U$, and A and B are dense.

We claim that A and B are connected. Let U and V be open sets meeting A such that $A \subset V \cup V$ and $U \cap V \cap A$ is void. Let $t \in U \cap A$ and $s \in V \cap A$. Let P be a subspace of Y containing s and t such that P is homeomorphic to the closed unit disc in R^2. Then either t is in the boundary of V, or $V \cap P$ is not dense in P and we can deduce from [N] that the boundary of $V \cap P$ contains C_β and a_β for some $\beta < \Gamma$. In any case A meets the boundary of V. But $A \setminus V \subset U$, so U meets V. Because A is dense in Y, the set $U \cap V \cap A$ must be nonvoid. This contradiction proves that A is connected. Likewise B is connected.

Finally, A and $Y \setminus A$ constitute a Bernstein parting of Y. □

Observe that any Euclidean space, or more generally any separable metric vector space, of dimension ≥ 2, satisfies the hypothesis of Lemma 1 and hence has a Bernstein parting. The same is true of any separable Banach space of dimension ≥ 2, or of the unit ball of such a Banach space. In particular, the Hilbert cube has a Bernstein parting.

We show that our space X satisfies the hypothesis of Lemma 1.

Lemma 2. Let A and B be two distinct elements of the space X. Then there is a subspace P of X containing A and B such that P is homeomorphic to the closed unit disc in R^2.

Proof. Let S be a measurable subset of E. If $m(S) > 0$, then for each real number t, $0 < t < 1$, there is the largest number $x(t)$ such that $m((-\infty, x(t)) \cap S) = t\, m(S)$. Put $(S)_t = (-\infty, x(t)) \cap S$ and $(S)_0 = \emptyset$, $(S)_1 = S$. If $m(S) = 0$, let $(S)_t = \emptyset$ for $0 \leq t < 1$ and $(S)_1 = S$. In any case, $(S)_t \subset (S)_q$ if $t \leq q$.

Either $m(A \setminus B) > 0$ or $m(B \setminus A) > 0$. Without less of generality, we assume $m(A \setminus B) > 0$. We use the work of the preceding paragraph to partition $A \setminus B$ into disjoint measurable subsets S_1, S_2 of equal measure; $S_1 \cup S_2 = A \setminus B$, and $m(S_1) = m(S_2) = m(A \setminus B)/2$. Lelt $S_3 = A \cap B$ and $S_4 = B \setminus A$. (Of course S_3 and S_4 might be null sets.) Let $(S_i)_t$ $(0 \leq t \leq 1)$ be as described in the preceding paragraph. Thus $m(S_i)_t = t\, m(S_i)$, $(S_i)_t \subset S_i$, and $(S_i)_t \subset (S_i)_q$ if $t \leq q$. Put $C = S_1$.

By direct computation we find that in R^2,

$$(m(A \cap S_1), m(A \cap S_2)) = (m(A \setminus B)/2, m(A \setminus B)/2),$$
$$(m(B \cap S_1), m(B \cap S_2)) = (0, 0),$$
$$(m(C \cap S_1), m(C \cap S_2)) = (m(A \setminus B)/2, 0),$$

and these points are the vertices of a triangle in R^2. Let T denote the 2-simplex of R^2 bounded by this triangle.

For $(u, v) \in T$, let (a, b, c) be the barycentric coordinates of (u, v); that is, $a \geq 0, b \geq 0, c \geq 0, a + b + c = 1$, and

$$(u, v) = a(m(A \cap S_1), m(A \cap S_2)) + b(m(B \cap S_1), m(B \cap S_2)) + c(m(C \cap S_1), m(C \cap S_2)).$$

Define $F(u, v) = (S_1)_{a+c} \cup (S_2)_a \cup (S_3)_{a+b} \cup (S_4)_b$. Then

$$F(m(A \cap S_1), m(A \cap S_2)) = S_1 \cup S_2 \cup S_3 = A,$$
$$F(m(B \cap S_1), m(B \cap S_2)) = S_3 \cup S_4 = B,$$
$$F(m(C \cap S_1), m(C \cap S_2)) = S_1 = C.$$

Observe that

$$\begin{aligned} m(F(u, v) \cap S_1) &= m(S_1)_{a+c} = am(S_1) + cm(S_1) \\ &= am(A \cap S_1) + bm(B \cap S_1) + cm(C \cap S_1) = u, \\ m(F(u, v) \cap S_2) &= m(S_2)_a = am(S_2) \\ &= am(A \cap S_2) + bm(B \cap S_2) + cm(C \cap S_2) = v. \end{aligned}$$

Thus F is a one-to-one mapping of T onto a subset $F(T)$ of X that contains the points A, B and C.

If (a', b', c') are the barycentric coordinates of another point $(u', v') \in T$, then the distance in X between the points $F(u, v)$ and $F(u', v')$ cannot exceed

$$\begin{aligned} &|m((S_1)_{a+c} \setminus (S_1)_{a'+c'})| + |m((S_1)_{a'+c'} \setminus (S_1)_{a+c})| + \\ &|m((S_2)_a \setminus (S_2)_{a'})| + |m((S_2)_{a'} \setminus (S_2)_a)| + \\ &|m((S_3)_{a+b} \setminus (S_3)_{a'+b'})| + |m((S_3)_{a'+b'} \setminus (S_3)_{a+b})| + \\ &|m((S_4)_b \setminus (S_4)_{b'})| + |m((S_4)_{b'} \setminus (S_4)_b)| \leq \\ &|a + c - a' - c'|m(S_1) + |a - a'|m(S_2) + |a + b - a' - b'|m(S_3) + |b - b'|m(S_4). \end{aligned}$$

It follows that F is a continuous one-to-one function mapping T into X. But T is compact, so F is a homeomorphism. Also $A, B \in F(T)$. Finally, T is also homeomorphic to the closed unit disc in R^2. □

Theorem. X has a Bernstein parting.

Proof. Lemmas 1 and 2. □

References

[HS] D.W. Hall & G.L Spencer II, Elementary Topology, John Wiley, New York, 1955, p. 147.

[N] M.A.H. Newman, Topology of Plane Sets, Cambridge University Press, 1961, Theorem 14.2, p. 123.

[SS] L.A. Steen & J.A. Seebach, Jr., Counterexamples in Topology, second edition, Springer-Verlag, New York, 1978, pp. 142, 225, 226.

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